Grade 12 – Book B

(First edition – CAPS)

TEACHER'S GUIDE

CONTENTS:

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WITH SPECIAL THANKS AND ACKNOWLEDGMENT TO THE DEPARTMENT OF EDUCATION FOR THE USE OF EXTRACTS FROM OLD PAPERS.

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REVISION FROM PAST PAPERS:

Exercise A: Date:____________________

(1) (a) Use the definition to differentiate $f(x) = 1 - 3x^2$. (Use first principles.) (4)

$$\rightarrow \; f(x+h) = 1 - 3(x+h)^2$$
$$= 1 - 3(x^2 - 2xh + h^2)$$
$$\therefore \; f(x+h) = 1 - 3x^2 + 6xh - 3h^2$$

But $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{(1 - 3x^2 + 6xh - 3h^2) - (1 - 3x^2)}{h}$ ✓

$= \lim_{h \to 0} \frac{1 - 3x^2 + 6xh - 3h^2 - 1 + 3x^2}{h}$

$= \lim_{h \to 0} \frac{6xh - 3h^2}{h}$ ✓

$= \lim_{h \to 0} \frac{h(6x - 3h)}{h}$ ✓

$= \lim_{h \to 0} (6x - 3h)$

$= 6x - 3(0)$

$\therefore f'(x) = 6x$ ✓

(b) Calculate $D_x\left[4 - \frac{4}{x^3} - \frac{1}{x^4}\right]$ (3)

$= D_x[4 - 4x^{-3} - 1x^{-4}]$ ✓

$= 12x^{-4} + 4x^{-5}$ ✓

$= \frac{12}{x^4} + \frac{4}{x^5}$ ✓

(c) Determine $\frac{dy}{dx}$ if $y = (1 + \sqrt{x})^2$ (3)

$\therefore \; y = 1 + 2\sqrt{x} + (\sqrt{x})^2$ ✓

$\therefore \; y = 1 + 2x^{\frac{1}{2}} + x$

$\therefore \; \frac{dy}{dx} = 2 \times \frac{1}{2} x^{-\frac{1}{2}} + 1$

$\therefore \; \frac{dy}{dx} = x^{\frac{1}{2}} + 1 = \frac{1}{x^{\frac{1}{2}}} + 1 = \frac{1}{\sqrt{x}} + 1$ ✓✓

(2) Given: $g(x) = (x - 6)(x - 3)(x + 2)$

(a) Calculate the y-intercept of g. $\rightarrow \; x = 0$ (1)

$\therefore \; g(0) = (0 - 6)(0 - 3)(0 + 2) = 36$

$\therefore \; (0\,;36)$ ✓

(b) Write down the x-intercepts of g. $\rightarrow \; y = 0$ (2)

$\therefore \; g(x) = 0 = (x - 6)(x - 3)(x + 2)$ ✓

$x = 6 \;\; ; \;\; x = 3 \;\;$ and $\;\; x = -2$

$\therefore \; (6\,;0) \;\; ; \;\; (3\,;0) \;\;$ and $\;\; (-2\,;0)$ ✓

(c) Determine the turning points of g. $\rightarrow$ $g'(x) = 0$ (6)

$g(x) = (x - 6)(x^2 - x - 6)$

$\therefore g(x) = x^3 - x^2 - 6x - 6x^2 + 6x + 36$

$\therefore g(x) = x^3 - 7x^2 + 36$ ✓

$\therefore g'(x) = 3x^2 - 14x = 0$ ✓✓

$\therefore x(3x - 14) = 0$

$\therefore x = 0$ or $x = \frac{14}{3} = 4\frac{2}{3} \approx 4,7$ ✓

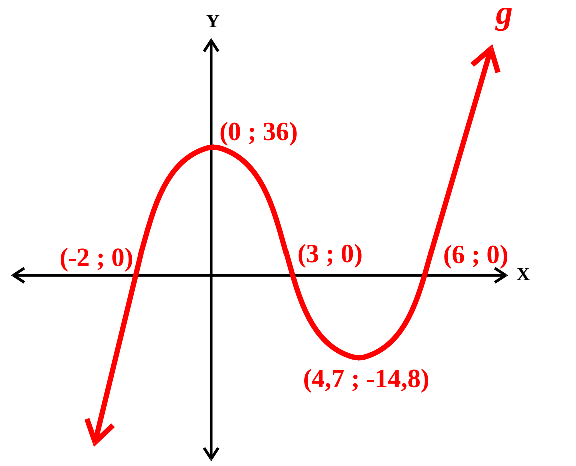
$\therefore (0\,;36)$ ✓

See y-intercept

$\therefore g\left(\frac{14}{3}\right) = \left(\frac{14}{3}\right)^3 - 7\left(\frac{14}{3}\right)^2 + 36 = -\frac{400}{27} \approx -14,8$

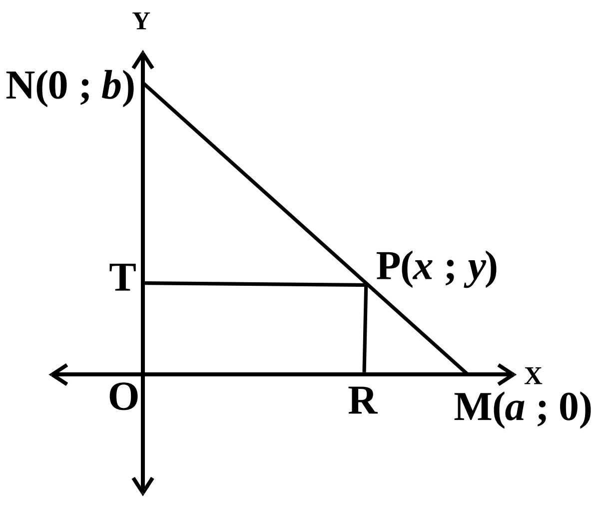
$\therefore \left(\frac{14}{3}\,; -\frac{400}{27}\right) = (4,7\,; -14,8)$ ✓

(d) Sketch the graph of g. (4)



✓✓ x-intercepte ✓ Turning points ✓ Shape

(3) A farmer has a piece of land in the shape of a right-angled triangle OMN, as shown in the figure below. He allocates a rectangular piece of land PTOR to his daughter, giving her the freedom to choose P anywhere along the boundary MN. Let OM = a, ON = b and P$(x\,;\,y)$ be any point on MN.



(a) Determine an equation of MN in terms of a and b. (2)

Gradient of MN $= \frac{b-0}{0-a} = -\frac{b}{a}$ ✓ with y-intercept at N $\rightarrow b$

$\therefore y = mx + c$

$\therefore y = -\frac{b}{a}x + b$ ✓

(b) Prove that the daughter's land will have a maximum area if she chooses P at the midpoint of MN. (6)

Area of PTOR = L × B = xy ✓ → See the coordinates of P

$\therefore \quad A(x) = x \times \left(-\frac{bx}{a} + b\right)$ ✓ → **Substitute y as determined in (a)**

$\therefore \quad A(x) = -\frac{b}{a}x^2 + bx$

$\therefore \quad A'(x) = -\frac{2b}{a}x + b = 0$ ✓✓ → **For maximum $A'(x) = 0$**

$\therefore \quad \frac{2b}{a}x = b$

$\therefore \quad x = \frac{b}{1} \times \frac{a}{2b}$

$\therefore \quad x = \frac{a}{2}$ ✓

Substitute x back in $y = -\frac{b}{a}x + b$ → $y = -\frac{b}{a} \times \frac{a}{2} + b$

$\therefore \quad y = -\frac{b}{2} + b = \frac{b}{2}$ ✓

$\therefore$ **P$\left(\frac{a}{2}\ ;\ \frac{b}{2}\right)$ which is the midpoint of MN.**

Exercise B: Date:____________________

(1) (a) Determine $f'(x)$ from first principles if $f(x) = 9 - x^2$ (5)

→ $f(x+h) = 9 - (x+h)^2$

→ $f(x+h) = 9 - (x^2 + 2xh + h^2)$

→ $f(x+h) = 9 - x^2 - 2xh - h^2$ ✓

But $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{(9 - x^2 - 2xh - h^2) - (9 - x^2)}{h}$ ✓

$= \lim_{h \to 0} \frac{9 - x^2 - 2xh - h^2 - 9 + x^2}{h}$

$= \lim_{h \to 0} \frac{-2xh - h^2}{h}$

$= \lim_{h \to 0} \frac{h(-2x - h)}{h}$

$= \lim_{h \to 0} (-2x - h)$ ✓

$= -2x - (0)$ ✓

$\therefore f'(x) = -2x$ ✓

(b) Evaluate:

(i) $D_x\left[1 + 6\sqrt{x}\right]$ (2)

$= D_x\left[1 + 6x^{\frac{1}{2}}\right]$ ✓

$= 6 \times \frac{1}{2}x^{-\frac{1}{2}}$

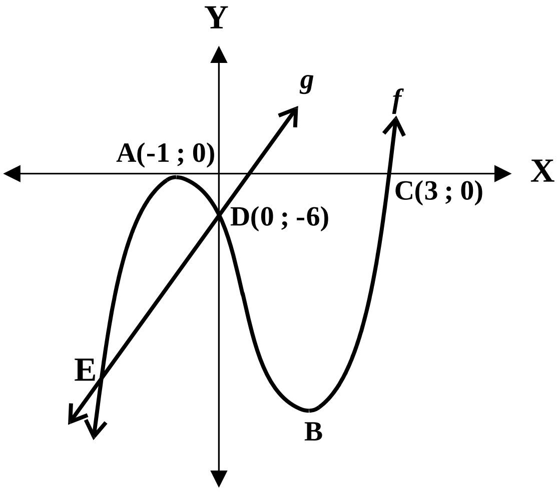
$= 3x^{-\frac{1}{2}} = \frac{3}{\sqrt{x}}$ ✓

(ii) $\frac{dy}{dx}$ if $y = \frac{8-3x^6}{8x^5}$ (4)

$\therefore\ y = \frac{8}{8x^5} - \frac{3x^6}{8x^5} = x^{-5} - \frac{3}{8}x$ ✓✓

$\therefore\ \frac{dy}{dx} = -5x^{-6} - \frac{3}{8} = \frac{-5}{x^6} - \frac{3}{8}$ ✓✓

(2) The graphs of $g(x) = 6x - 6$ and $f(x) = ax^3 + bx^2 + cx + d$ are sketched alongside. A$(-1\,;\,0)$ and C$(3\,;\,0)$ are the x-intercepts of f. The graph of f has turning points at A and B. D$(0\,;\,-6)$ is the y-intercept of f. E and D are the points of intersection of the graphs of f and g.



(a) Show that $a = 2$; $b = -2$; $c = -10$ and $d = -6$. (5)

$f(x) = a(x+1)(x+1)(x-3)$ ✓✓

$\therefore\ f(x) = a(x^2+2x+1)(x-3)$

$\therefore\ f(x) = a(x^3 - 3x^2 + 2x^2 - 6x + 1x - 3)$

$\therefore\ -6 = a((0)^3 - (0)^2 - 5(0) - 3)$ ✓ through D$(0\,;-6)$ (x y)

$\therefore\ -6 = a(-3)$

$\therefore\ a = \frac{-6}{-3} = 2$ ✓

$\therefore\ f(x) = 2(x^3 - x^2 - 5x - 3)$

$\therefore\ f(x) = 2x^3 - 2x^2 - 10x - 6$ ✓

$\therefore\ a = 2\ ;\ b = -2\ ;\ c = -10$ and $d = -6.$

(b) Calculate the coordinates of the turning point B. $\rightarrow$ $f'(x) = 0$ (5)

$f'(x) = 6x^2 - 4x - 10 = 0$ ✓✓

$\therefore\ 3x^2 - 2x - 5 = 0$

$\therefore\ (3x-5)(x+1) = 0$

$\therefore\ x = \frac{5}{3} \approx 1,67$ or $x = -1$ ✓

$\therefore\ y = f\left(\frac{5}{3}\right) = 2\left(\frac{5}{3}\right)^3 - 2\left(\frac{5}{3}\right)^2 - 10\left(\frac{5}{3}\right) - 6$

$\therefore\ y = \frac{-512}{27} \approx -18,96$

$\therefore$ B$\left(\frac{5}{3}; \frac{-52}{27}\right) = (1,67\,; -18,96)$ ✓✓

(c) $h(x)$ is the vertical distance between $f(x)$ and $g(x)$, that is, $h(x) = f(x) - g(x)$. Calculate x such that $h(x)$ is a maximum, where $x < 0$. (5)

$\therefore\ h(x) = (2x^3 - 2x^2 - 10x - 6) - (6x - 6)$

$\therefore\ h(x) = 2x^3 - 2x^2 - 10x - 6 - 6x + 6$

$\therefore\ h(x) = 2x^3 - 2x^2 - 16x$ ✓

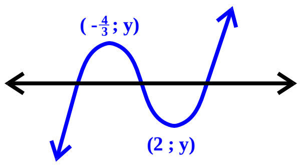
For maximum $\rightarrow$ $h'(x) = 0$ ✓

✓ $\therefore\ h'(x) = 6x^2 - 4x - 16 = 0$

$\therefore\ 3x^2 - 2x - 8 = 0$

$\therefore\ (3x + 4)(x - 2) = 0$ ✓

$\therefore\ x = -\frac{4}{3}$ ✓ **or** $x = 2$



(3) The tangent to the curve of $g(x) = 2x^3 + px^2 + qx - 7$ at $x = 1$ has the equation of $y = 5x - 8$.

(a) Show that $(1\ ;\ -3)$ is the point of contact of the tangent to the graph. (1)

$x = 1$ **given** $\rightarrow$ $y = 5(1) - 8 = 5 - 8 = -3$ ✓

$\therefore$ **Point of contact is** $(1\ ;\ -3)$

(b) Hence or otherwise, calculate the values of p and q. (6)

$-3 = g(1) = 2(1)^3 + p(1)^2 + q(1) - 7$

$\therefore\ -3 = 2 + p + q - 7$ ✓ $\rightarrow$ $-3 = p + q - 5$

$\therefore\ p = 2 - q$ .... (1) ✓

But $m = g'(x) = 2x^3 + px^2 + qx - 7$ **with** $m = 5$ $\rightarrow$ **See tangent**

$\therefore\ 5 = g'(1) = 6(1)^2 + 2p(1) + q$ ✓

$\therefore\ 5 = 6 + 2p + q$ ...... (2)

Substitute (1) in (2): $5 = 6 + 2(2 - q) + q$ ✓

$\therefore\ 5 = 6 + 4 - 2q + q$

$\therefore\ 5 = 10 - q$

$\therefore\ q = 5$ ✓ $\rightarrow$ $p = 2 - 5$ $\therefore\ p = -3$ ✓

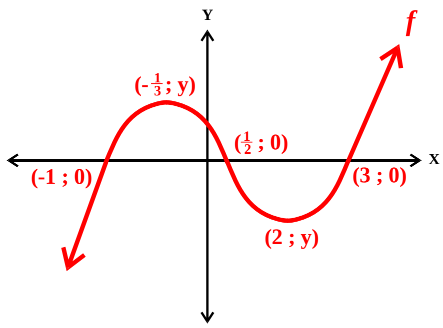
(4) A cubic function f has the following properties: (4)

* $f\left(\frac{1}{2}\right) = f(3) = f(-1) = 0$

* $f'(2) = f'\left(-\frac{1}{3}\right) = 0$

* f decreases for $x \in \left[-\frac{1}{3}\ ;2\right]$

Draw a possible sketch graph of f, clearly indicating the x-coordinates of the turning points and ALL the x-intercepts.



✓ x-int ✓ Turning point ✓✓ Shape

Exercise C: Date:____________________

(1) (a) Determine $f'(x)$ from first principles if $f(x) = 2x^2 - 5$ (5)

$\rightarrow f(x+h) = 2(x+h)^2 - 5$ ✓

$\rightarrow f(x+h) = 2(x^2 + 2xh + h^2) - 5$

$\rightarrow f(x+h) = 2x^2 + 4xh + 2h^2 - 5$ ✓

But $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{(2x^2 + 4xh + 2h^2 - 5) - (2x^2 - 5)}{h}$ ✓

$= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 5 - 2x^2 + 5}{h}$

$= \lim_{h\to 0} \frac{4xh + 2h^2}{h}$

$= \lim_{h\to 0} \frac{h(4x + 2h)}{h}$

$= \lim_{h\to 0} (4x + 2h)$ ✓

$= 4x - 2(0)$

$\therefore f'(x) = 4x$ ✓

(b) Evaluate: $\frac{dy}{dx}$ if $y = x^{-4} + 2x^3 - \frac{x}{5}$ (3)

$\therefore \frac{dy}{dx} = -4x^{-5} + 6x^2 - \frac{1}{5} = \frac{-4}{x^5} + 6x^2 - \frac{1}{5}$
✓ ✓ ✓

(c) Given: $g(x) = \frac{x^2 + x - 2}{x - 1}$

(i) Calculate $g'(x)$ for $x \neq 1$. (2)

$g(x) = \frac{(x+2)(x-1)}{(x-1)} = (x+2)$ ✓ $g'(x) = 1$ ✓

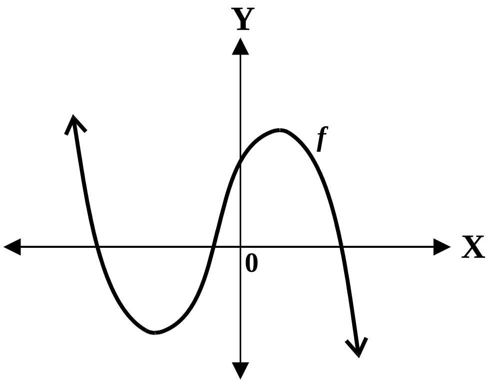
(ii) Explain why it is not possible to determine $g'(1)$. (1)

If $x = 1$, then the denominator will be 0 and that is undefined. ✓

(2) The graph of the function

$f(x) = -x^3 - x^2 + 16x + 16$

is sketched alongside.



(a) Calculate the x-coordinates of the turning points of f. (4)

Turning points $\rightarrow f'(x) = 0$

$f'(x) = -3x^2 - 2x + 16 = 0$ ✓✓

$\therefore 3x^2 + 2x - 16 = 0$

$\therefore (3x + 8)(x - 2) = 0$ ✓

$\therefore x = -\frac{8}{3}$ or $x = 2$ ✓

(b) Calculate the x-coordinate of the point at which $f'(x)$ is a maximum. (3)

Maximum where $f''(x) = 0$

$\therefore \ f''(x) = -6x - 2 = 0$ ✓✓

$\therefore \quad x = -\frac{1}{3}$ ✓

(3) Consider the graph of $g(x) = -2x^2 - 9x + 5$.

(a) Determine the equation of the tangent to the graph of g at $x = -1$. (4)

$\therefore \quad g(-1) = -2(-1)^2 - 9(-1) + 5 = -2 + 9 + 5 = 12$

$\therefore$ **Point of contact:** $(-1\,;12)$ ✓

$m = g'(x) = -4x - 9$ ✓

$\therefore \quad m = g'(-1) = -4(-1) - 9 = 4 - 9 = -5$ ✓

With $\quad y - y_1 = m(x - x_1)$

$\therefore \quad y - 12 = -5(x - (-1))$

$\therefore \quad y = -5x - 5 + 12$

$\therefore \quad y = -5x + 7$ ✓

(b) For which values of q will the line $y = -5x + q$ not intersect the parabola? (3)

Equal: $\quad -5x + q = -2x^2 - 9x + 5$

$\therefore \quad 0 = -2x^2 - 9x + 5 + 5x - q$

$\therefore \quad 2x^2 + 4x + q - 5 = 0$ ✓

But $\quad \Delta = b^2 - 4ac$

$\therefore \quad \Delta = (4)^2 - 4(2)(q - 5) = 16 - 8q + 40$

Not intersect $\rightarrow \quad \Delta < 0$

$\therefore \quad 16 - 8q + 40 < 0$ ✓

$\therefore \quad -8q < -56$

$\therefore \quad q > 7$ ✓

(4) Given: $h(x) = 4x^3 + 5x$

Explain whether it is possible to draw a tangent to the graph of h that has a negative gradient. Show ALL your calculations. (3)

$m = h'(x) = 12x^2 + 5$ ✓

but $\quad x^2 \geq 0$

$\therefore \quad 12x^2 \geq 0$

$\therefore \quad 12x^2 + 5 \geq 0 + 5$

$\therefore \quad 12x^2 + 5 \geq 5$ ✓

$\therefore$ **The gradient of the tangent will always be greater than 5.**

$\therefore$ **The gradient of the tangent will never be negative.** ✓

(5) A particle moves along a straight line. The distance, s (in metres) of the particle from a fixed point on the line at time t seconds ($t \geq 0$) is given by
$s(t) = 2t^2 - 18t + 45$.

(a) Calculate the particle's initial velocity. (Velocity is the rate of change of distance.)

$s(t) = 2t^2 - 18t + 45$

$\therefore\ s'(t) = 4t - 18$ ✓ → Velocity

$\therefore\ s'(0) = 4(0) - 18$ ✓ → No time lapse

$\therefore\ s'(t) = -18\ \text{m/s}$ ✓

(b) Determine the rate at which the velocity of the particle is changing at t seconds.

$\therefore\ s''(t) = 4\ \text{m/s}^2$ ✓

(c) After how many seconds will the particle be closest to the fixed point?

$\therefore\ s'(t) = 4t - 18 = 0$ ✓ → Distance is a minimum

$\therefore\ 4t = 18$

$\therefore\ t = \frac{18}{4} = 4{,}5\ \text{sec}$ ✓

Exercise D: Date:____________________

(1) (a) Use the definition of the derivative (first principles) to determine $f'(x)$ if $f(x) = 2x^2$. (5)

→ $f(x+h) = 2(x+h)^2$

→ $f(x+h) = 2(x^2 + 2xh + h^2)$

→ $f(x+h) = 2x^2 + 4xh + 2h^2$ ✓

But $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ ✓

$= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 2x^2}{h}$

$= \lim_{h \to 0} \frac{4xh + 2h^2}{h}$ ✓

$= \lim_{h \to 0} \frac{h(4x + 2h)}{h}$

$= \lim_{h \to 0} (4x + 2h)$ ✓

$= 4x - 2(0)$

$\therefore\ f'(x) = 4x$ ✓

(b) Determine $\frac{dy}{dx}$ if $y = \frac{2\sqrt{x} + 1}{x^2}$ (4)

$\therefore\ y = \frac{2x^{\frac{1}{2}}}{x^2} + \frac{1}{x^2}$

$\therefore\ y = 2x^{-1{,}5} + x^{-2}$ ✓✓

$\therefore\ \frac{dy}{dx} = 2 \times -1{,}5x^{-2{,}5} - 2x^{-3}$

$\therefore\ \frac{dy}{dx} = -3x^{-2{,}5} - 2x^{-3} = -\frac{3}{x^{2{,}5}} - \frac{2}{x^3}$ ✓✓

(c) Calculate the values of a and b if $f(x) = ax^2 + bx + 5$ has a tangent at $x = -1$ which is defined by the equation $y = -7x + 3$. (6)

$y = -7(-1) + 3 = 7 + 3 = 10 \rightarrow$ **Point of contact:** $(-1\,;10)$ ✓

$\therefore \quad y = 10 = f(-1) = a(-1)^2 + b(-1) + 5$

$\therefore \quad 10 = a - b + 5$

$\therefore \quad b = a + 5 - 10$ ✓

$\therefore \quad b = a - 5 \quad \ldots\ldots\ (1)$

But $m = f'(x) = 2ax + b$ ✓ **with** $m = -7 \rightarrow$ **See tangent**

$\therefore \quad -7 = f'(-1) = 2a(-1) + b$ ✓

$\therefore \quad -7 = -2a + b \quad \ldots\ldots\ (2)$

Substitute (1) in (2): $-7 = -2a + a - 5 = -a + 5$

$\therefore \quad a = -5 + 7 = 2$ ✓

$\therefore \quad b = 2 - 5 = -3$ ✓

(2) Given: $f(x) = -x^3 - x^2 + x + 10$

(a) Write down the coordinates of the y-intercept of f. $\rightarrow$ $(0\,;10)$ ✓ (1)

(b) Show that $(2\,;\ 0)$ is the only x-intercept of f. $\rightarrow$ $x = 0$ (4)

$\therefore \quad 0 = -x^3 - x^2 + x + 10$

$\therefore \quad 0 = x^3 + x^2 - x - 10$ **with** $x = 2$ **given** $\rightarrow$ $(x-2)$ **is a factor** ✓

$\therefore \quad 0 = (x-2)(x^2 + 3x + 5)$ ✓ $\rightarrow$ **through inspection**

$\therefore \quad x = 2$ **or** $x^2 + 3x + 5 = 0$

$\rightarrow$ **only x-intercept**

$\therefore \quad \Delta = b^2 - 4ac$

$= (3)^2 - 4(1)(5) = 9 - 20$

$\therefore \quad \Delta = -11$ ✓

$\therefore$ **No real roots** ✓

(c) Calculate the coordinates of the turning points of f. $\rightarrow$ $f'(x) = 0$ (6)

$f'(x) = -3x^2 - 2x + 1 = 0$ ✓✓

$\therefore \quad 3x^2 + 2x - 1 = 0$

$\therefore \quad (x+1)(3x-1) = 0$ ✓

$\therefore \quad x = -1$ **or** ✓ $x = \frac{1}{3}$

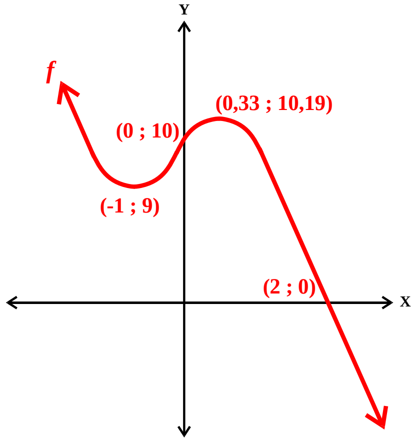
$\therefore f(-1) = -(-1)^3 - (-1)^2 + (-1) + 10$

$\therefore f(-1) = 9 \rightarrow (-1\,;9)$ ✓

$f\left(\frac{1}{3}\right) = -\left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right) + 10$

$f\left(\frac{1}{3}\right) = \frac{275}{27} \rightarrow \left(\frac{1}{3}\,;\ \frac{275}{27}\right)$ ✓

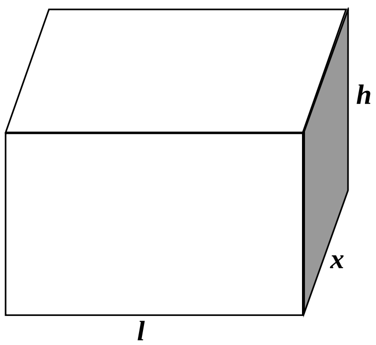
(d) Sketch the graph of f. Show all intercepts with the axes and all turning points. (3)



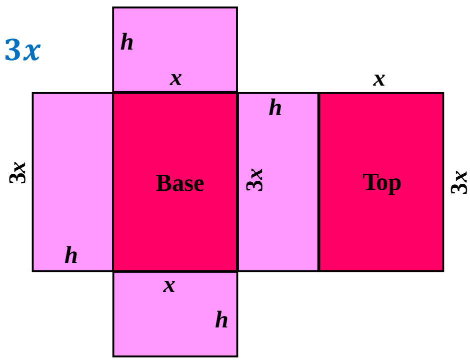
✓ x-int ✓ Turning p ✓ Shape

(3) A rectangular box is constructed in such a way that the length (l) of the base is three times as long as its width. The material used to construct the top and the bottom of the box costs R100 per square metre. The material used to construct the sides of the box costs R50 per square metre. The box must have a volume of 9 m^3. Let the width of the box be x metres.

A rectangular box is constructed in such a way that the length (l) of the base is three



$l = 3b = 3x$



(a) Determine an expression for the height (h) of the box in terms of x. (3)

Volume = L × B × H

$\therefore\ 9 = 3x \times x \times h$ → **Length is given as three times the width** ✓

$\therefore\ 9 = 3x^2 \times h$ ✓

$\therefore\ h = \frac{9}{3x^2} = \frac{3}{x^2}$ ✓

(b) Show that the cost to construct the box can be expressed as $K = \frac{1\,200}{x} + 600x^2$. (3)

SA = Base + Top + 4 sides

$SA = [3x \times x + 3x \times x] + [2 \times 3xh + 2 \times xh]$ ✓ → **See diagram above!**

$\therefore\ K = [2 \times 3x^2] \times 100 + [6xh + 2xh] \times 50$

$\therefore\ K = 600x^2 + 400xh$ ✓

$\therefore\ K = 600x^2 + 400x \times \left(\frac{3}{x^2}\right)$ ✓ → **See (a)**

$\therefore\ K = 600x^2 + \frac{400x \times 3}{x^2}$

$\therefore\ K = 600x^2 + \frac{1\,200}{x}$