Grade 11 – Book C TG

(CAPS Edition)

CONTENT:

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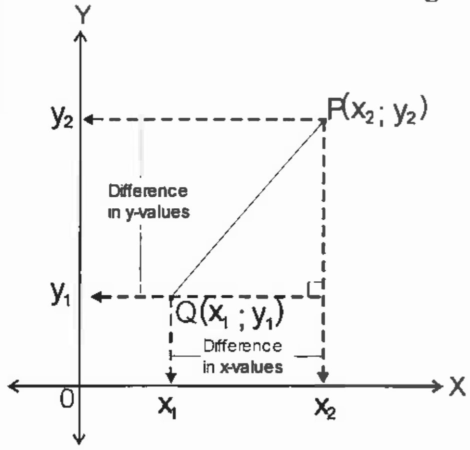
Chapter C1

Analytical geometry

C1.1 Gradient:
C1.1.1 Calculating the gradient:

In grade 10 the following formula for the gradient of the straight line were derived:

Deduction of a formula for the gradient of a straight line:

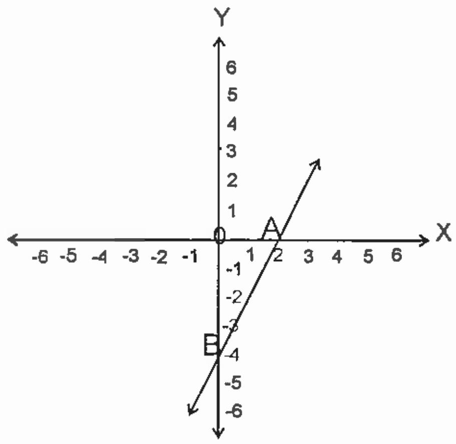


$$\text{Gradient/Inclination} = \frac{\text{Difference in } y \text{ values}}{\text{Difference in } x \text{ values}}$$

$$m_{PQ} = \frac{\text{Difference in } y \text{ values}}{\text{Difference in } x \text{ values}}$$

$$m_{PQ} = \underline{\frac{y_2 - y_1}{x_2 - x_1}}$$

E.g.1



In the graph on the left, the straight line is through A(2 ; 0) and B(0 ; -4).
The difference between the y-values is:
- 4 - 0 = - 4 and
The difference between the x-values is:
0 - 2 = - 2

$$\therefore \text{ gradient } = \frac{\text{difference between } y\text{-values}}{\text{difference between } x\text{-values}}$$

$$= \frac{-4}{-2}$$

$$m_{AB} = \underline{2}$$

E.g.2 Calculate the gradient of the line through the following points: M(2 ; -1) and N(-2 ; 3)

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$$\begin{matrix} x_1 & y_1 & & x_2 & y_2 \end{matrix}$$

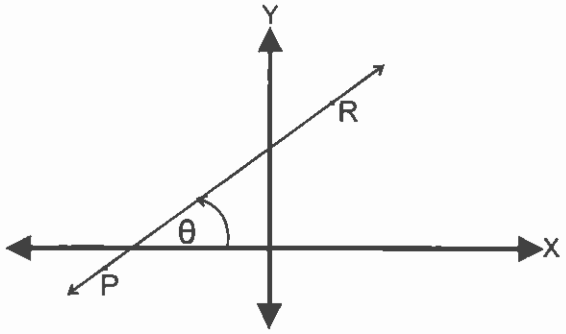
M(2 ; -1) and N(-2 ; 3)

$$m_{MN} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-1)}{-2 - (2)} = \frac{3 + 1}{-2 - 2} = \frac{4}{-4}$$

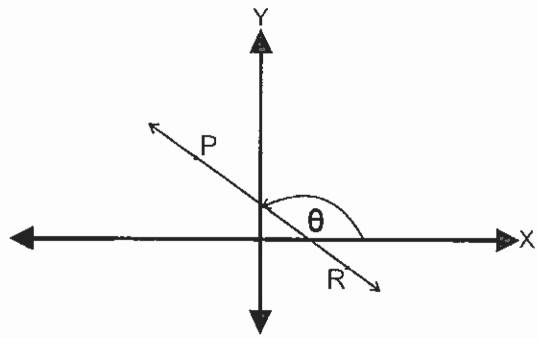
$$\therefore m_{MN} = \underline{-1}$$

C1.1.2 Application of the gradient:

* Parallel lines have the same gradients: If $m_1 = m_2 \Leftrightarrow$ the lines are parallel.
* The product of the gradients of perpendicular lines is equal to -1 : If $m_1 \times m_2 = -1 \Leftrightarrow$ the lines are perpendicular.
* Three or more points are collinear if the points lie on the same straight line.
  $\therefore\ m_{AB} = m_{BC} \Leftrightarrow$ points A, B and C lies on the same straight line.
* The angle of inclination is the angle between the straight line and the positive x-axis:



The angle of inclination above is θ and it is an acute angle ($0° < \theta < 90°$), if the line has a positive gradient.



The angle of inclination above is θ and it is an obtuse angle ($90° < \theta < 180°$), if the line has a negative gradient.

To calculate the angle of inclination: $\tan\theta = m_{PR}$

E.g.3 Consider: P(-3 ; - 2) , Q(5 ; 4) and R(1 ; -4)
(a) Determine whether the points are collinear.
(b) Prove that $QR \perp PR$.
(c) Calculate the angle of inclination (correct to 2 decimals) of line PQ.

****

(a) $$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{4-(-2)}{5-(-3)} = \frac{4+2}{5+3} = \frac{6}{8} = \frac{3}{4}$$

$$m_{QR} = \frac{y_R - y_Q}{x_R - x_Q} = \frac{-4-4}{1-5} = \frac{-8}{-4} = 2$$

$\therefore$ *P, Q and R is not collinear, because* $m_{PQ} \neq m_{QR}$

(b) *We calculated in (a) that* $m_{QR} = 2$

$$m_{PR} = \frac{y_R - y_P}{x_R - x_P} = \frac{-4-(-2)}{1-(-3)} = \frac{-4+2}{1+3} = \frac{-2}{4} = \frac{-1}{2}$$

$$\therefore\ m_{QR} \times m_{PR} = \frac{2}{1} \times \frac{-1}{2} = -1$$

$\therefore\ QR \perp PR$

(c) *We calculated in (a) that* $m_{PQ} = \frac{3}{4}$

$$\therefore\ \tan\theta = \frac{3}{4}$$

$$\therefore\ \underline{\theta = 36{,}87°}$$

Exercise 1: Date:______________

(1) Determine whether points A, B and C are collinear or not:

(a) A(1 ; 2) , B(3 ; 5) and C(5 ; 7)

$m_{AB} = \frac{y_B - y_A}{x_B - x_A}$

$= \frac{5-2}{3-1} = \frac{3}{2}$

$m_{BC} = \frac{y_C - y_B}{x_C - x_B}$

$= \frac{7-5}{5-3} = \frac{2}{2} = 1$

∴ A, B and C not collinear!

(b) A(-1 ; 3) , B(4 ; 0) and C(14 ; 6)

$m_{AB} = \frac{y_B - y_A}{x_B - x_A}$

$= \frac{0-3}{4-(-1)} = \frac{-3}{5}$

$m_{BC} = \frac{y_C - y_B}{x_C - x_B}$

$= \frac{6-0}{14-4} = \frac{6}{10}$

∴ A, B and C not collinear!

(2) M(-2 ; -4) , N(1 ; -3) , R(2 ; -1) , T(-3 ; -1) and K(3 ; -4)

(a) Determine which of the following lines are parallel and which are perpendicular: MN, TK, RK , NR and TM

(b) Without calculating the angle of inclination, determine which of the lines in (a) have an acute angle of inclination.

(c) Calculate the angle of inclination of line TN.

(a) $m_{MN} = \frac{y_N - y_M}{x_N - x_M} = \frac{-3-(-4)}{1-(-2)} = \frac{-3+4}{1+2} = \frac{1}{3}$

$m_{TK} = \frac{y_K - y_T}{x_K - x_T} = \frac{-4-(-1)}{3-(-3)} = \frac{-4+1}{3+3} = \frac{-3}{6} = \frac{-1}{2}$

$m_{RK} = \frac{y_K - y_R}{x_K - x_R} = \frac{-4-(-1)}{3-2} = \frac{-4+1}{3-2} = \frac{-3}{1} = -3$

$m_{NR} = \frac{y_R - y_N}{x_R - x_N} = \frac{-1-(-3)}{2-1} = \frac{-1+3}{2-1} = \frac{2}{1} = 2$

$m_{TM} = \frac{y_M - y_T}{x_M - x_T} = \frac{-4-(-1)}{-2-(-3)} = \frac{-4+1}{-2+3} = \frac{-3}{1} = -3$

∴ Parallel lines: RK // TM Perpendicular lines: MN ⊥ RK, TK ⊥ NR.

(b) Lines MN and NR (positive gradients!)

(c) $m_{TN} = \frac{y_N - y_T}{x_N - x_T}$

$= \frac{-3-(-1)}{1-(-3)}$

$= \frac{-3+1}{1+3} = \frac{-2}{4} = -\frac{1}{2}$

∴ $\tan\theta = m = -\frac{1}{2}$

∴ $\theta = 180° - 26{,}6°$

$\theta = 153{,}4°$

(3) D(-3 ; -1) , E(0 ; -4) , F(-1 ; y) , G(x ; 3) and H(2 ; 2). Calculate the value of:

(a) x, if EG // DH

$m_{EG} = \frac{y_G - y_E}{x_G - x_E} = \frac{3-(-4)}{x-0}$

$m_{DH} = \frac{y_H - y_D}{x_H - x_D} = \frac{2-(-1)}{2-(-3)} = \frac{3}{5}$

∴ $\frac{3+4}{x-0} = \frac{3}{5}$ (EG // DH)

$\frac{7}{x} = \frac{3}{5}$

$35 = 3x$

$x = \frac{35}{3} = 11\frac{2}{3}$

(b) y, if FH ⊥ DE

$m_{FH} = \frac{y_H - y_F}{x_H - x_F} = \frac{2-y}{2-(-1)} = \frac{2-y}{3}$

$m_{DE} = \frac{y_E - y_D}{x_E - x_D} = \frac{-4-(-1)}{0-(-3)} = \frac{-4+1}{0+3} = \frac{-3}{3}$

∴ $\frac{2-y}{3} \times \frac{-3}{3} = -1$ (FH ⊥ DE)

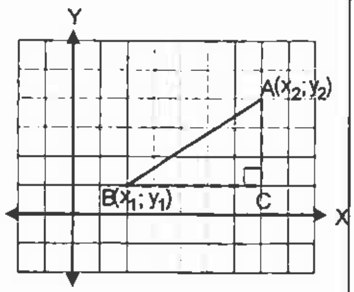
$(2-y)(-1) = (-1)(3)$

$-2+y = -3$

$y = -1$

C1.2 Distance between two points:

Derivation of a formula for the distance between any two coordinates:



The coordinates of C will be $(x_2\ ; y_1)$ because A and C have the same x-coordinates and B and C have the same y-coordinates.
The length of BC is the difference between the two x-coordinates of B and C and the length of AC is the difference between the y-coordinates of A and C.

$\therefore$ BC = $x_2 - x_1$ and AC = $y_2 - y_1$ [Remember: BC = CB!]

$\therefore$ $AB^2 = BC^2 + AC^2$ [Pythagoras]

$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$\sqrt{AB^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d(AB) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

E.g.4 Calculate the distance between S(7 ; -5) and T(4 ; -2). If necessary, write your answer as a simple surd.

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$x_1\ y_1 \qquad x_2\ y_2$

S(7 ; -5) and T(4 ; -2)

$\therefore\ d(ST) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d(ST) = \sqrt{[(4)-(7)]^2 + [(-2) - (-5)]^2}$

$d(ST) = \sqrt{(4 - 7)^2 + (-2 + 5)^2}$

$d(ST) = \sqrt{(-3)^2 + (3)^2}$

$d(ST) = \sqrt{9 + 9}$

$d(ST) = \sqrt{18}$

$d(ST) = \sqrt{9 \times 2}$

$d(ST) = 3\sqrt{2}$

Exercise 2: Date:_______________

(1) Calculate the distance between P and Q in each of the following. If necessary, round off, correct to two decimals:

(a) P(2 ; 5) and Q(7 ; 4)

$d(PQ) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$

$= \sqrt{(2-7)^2 + (5-4)^2}$

$= \sqrt{(-5)^2 + (1)^2}$

$= \sqrt{25 + 1}$

$= \sqrt{26}$

$d(PQ) \approx 5{,}10$

(b) P(-2 ; -1) and Q(0 ; 5)

$d(PQ) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$

$= \sqrt{(-2-0)^2 + (-1-5)^2}$

$= \sqrt{(-2)^2 + (-6)^2}$

$= \sqrt{4 + 36}$

$= \sqrt{40}$

$d(PQ) \approx 6{,}32$

(c) P(-3 ; 1) and Q(-3 ; 13)

$d(PQ) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$

$= \sqrt{(-3-(-3))^2 + (1-13)^2}$

$= \sqrt{(0)^2 + (-12)^2}$

$= \sqrt{0 + 144}$

$= \sqrt{144}$

$d(PQ) = 12$

(d) P(2,3 ; 3,1) and Q(5,3 ; 1,1)

$d(PQ) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$

$= \sqrt{(2{,}3 - 5{,}3)^2 + (3{,}1 - 1{,}1)^2}$

$= \sqrt{(-3)^2 + (2)^2}$

$= \sqrt{9+4}$

$= \sqrt{13}$

$d(PQ) \approx 3{,}61$

(e) P(2m ; m) and Q(7m ; - 4m)

$d(PQ) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$

$= \sqrt{(2m - 7m)^2 + (m - (-4m))^2}$

$= \sqrt{(-5m)^2 + (m+4m)^2}$

$= \sqrt{25m^2 + (5m)^2}$

$= \sqrt{25m^2 + 25m^2}$

$= \sqrt{50m^2}$

$d(PQ) \approx 7{,}07m$

(2) Calculate d(AB) in each of the following. Where necessary, leave your answer as a simple surd.

(a) A(1 ; $\sqrt{8}$) and B(-7 ; 0)

$d(AB) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$

$= \sqrt{(1-(-7))^2 + (\sqrt{8} - 0)^2}$

$= \sqrt{(1+7)^2 + (\sqrt{8})^2}$

$= \sqrt{(8)^2 + 8}$

$= \sqrt{64 + 8}$

$= \sqrt{72} = \sqrt{36 \times 2}$

$d(AB) = 6\sqrt{2}$

(b) A(-10 ; 9) and B(-2 ; 15)

$d(AB) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$

$= \sqrt{(-10-(-2))^2 + (9-15)^2}$

$= \sqrt{(-10+2)^2 + (-6)^2}$

$= \sqrt{(-8)^2 + 36}$

$= \sqrt{64 + 36}$

$= \sqrt{100}$

$d(AB) = 10$

(c) A(4 ; 1) and B(-4 ; 9)

$d(AB) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$

$= \sqrt{(4-(-4))^2 + (1-9)^2}$

$= \sqrt{(4+4)^2 + (-8)^2}$

$= \sqrt{(8)^2 + 64}$

$= \sqrt{64 + 64}$

$= \sqrt{128} = \sqrt{64 \times 2}$

$d(AB) = 8\sqrt{2}$

(3) Calculate the value(s) of p if d(LM) = 5 with L(-2 ; p) and M(-5 ; 3).

$d(LM) = \sqrt{(x_L - x_M)^2 + (y_L - y_M)^2}$

$5 = \sqrt{(-2-(-5))^2 + (p-3)^2}$

$(5)^2 = \left(\sqrt{(-2+5)^2 + (p-3)^2}\right)^2$

$25 = (3)^2 + (p-3)^2$

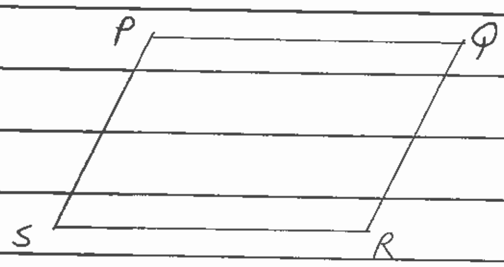
$25 = 9 + p^2 - 6p + 9$

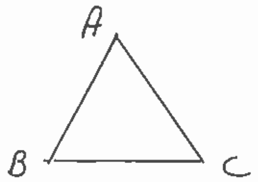
$0 = p^2 - 6p - 7$

$0 = (p-7)(p+1)$

$p = 7$ or $p = -1$

(5)





(4) A(2 ; -2) , B(3 ; 4) and C(-3 ; 5) is the vertices of triangle ABC.

(a) Calculate the perimeter of triangle ABC, correct to 1 decimal.

$* \ d(AB) = \sqrt{(3-2)^2 + (4-(-2))^2} = \sqrt{(1)^2 + (6)^2} = \sqrt{1+36}$

$= \sqrt{37} \approx 6{,}08\ldots$

$* \ d(BC) = \sqrt{(-3-3)^2 + (5-4)^2} = \sqrt{(-6)^2 + (1)^2} = \sqrt{36+1}$

$= \sqrt{37} \approx 6{,}08\ldots$

$* \ d(AC) = \sqrt{(-3-2)^2 + (5-(-2))^2} = \sqrt{(-5)^2 + (7)^2} = \sqrt{25+49}$

$= \sqrt{74} \approx 8{,}60\ldots$

$\therefore$ Perimeter $= \sqrt{37} + \sqrt{37} + \sqrt{74} = 20{,}8$

(b) Prove that $\hat{B} = 90°$.

$AB^2 + BC^2$

$= (\sqrt{37})^2 + (\sqrt{37})^2$

$= 37 + 37$

$= 74$

$AC^2 = (\sqrt{74})^2 = 74$

$\therefore AC^2 = AB^2 + BC^2$

$\therefore$ Pythagoras applicable

$\therefore \hat{B} = 90°$

(5) P(-2 ; 0) , Q(-1 ; -3) , R(2 ; 0) and S(1 ; 3) is the vertices of a parallelogram. Draw a diagram!

(a) Determine whether PQRS is a rhombus or not.

$d(PQ)$

$= \sqrt{(-2-(-1))^2 + (0-(-3))^2}$

$= \sqrt{(-1)^2 + (3)^2}$

$= \sqrt{1+9}$

$= \sqrt{10}$

$d(QR)$

$= \sqrt{(-1-2)^2 + (-3-0)^2}$

$= \sqrt{(-3)^2 + (-3)^2}$

$= \sqrt{9+9}$

$= \sqrt{18}$

PQRS isn't a rhombus, because adjacent sides not equal!

(b) Calculate the gradient of PS:

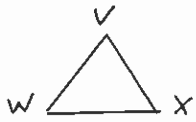
$m_{PS} = \frac{y_2 - y_1}{x_2 - x_1}$ $\quad P(-2;0)$ with $x_1\ y_1$ $\quad S(1;3)$ with $x_2\ y_2$

$= \frac{3-0}{1-(-2)}$

$m_{PS} = \frac{3}{1+2} = \frac{3}{3} = 1$

(c) Without any further calculations, determine the gradient of QR. Motivate your answer.

$m_{QR} = 1$, opp. sides of parm //, $\therefore$ gradients are the same.



(6) Determine whether ΔVWX is an isosceles or an equilateral triangle with V(2 ; 6) , W(3 ; -1) and X(-3 ; 1). Show all calculations:

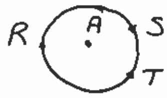
$d(VW) = \sqrt{(2-3)^2 + (6-(-1))^2}$
$= \sqrt{(-1)^2 + (7)^2}$
$= \sqrt{1+49} = \sqrt{50}$

$d(WX) = \sqrt{(3-(-3))^2 + (-1-1)^2}$
$= \sqrt{(6)^2 + (-2)^2}$
$= \sqrt{36+4} = \sqrt{40}$

$d(VX) = \sqrt{(2-(-3))^2 + (6-1)^2}$
$= \sqrt{(5)^2 + (5)^2}$
$= \sqrt{25+25} = \sqrt{50}$

$\therefore VW = VX \neq WX$ $\therefore$ ΔVWX is an isosceles Δ !

(7) S(-2 ; 3), T(1 ; 2) and R(-3 ; 0) are three points around the point A(-1 ; 1). Show that S, T and R are points on the circumference of the circle with A as midpoint.



$d(AS) = \sqrt{(-2-(-1))^2 + (3-1)^2}$
$= \sqrt{(-1)^2 + (2)^2}$
$= \sqrt{1+4} = \sqrt{5}$

$d(AT) = \sqrt{(1-(-1))^2 + (2-1)^2}$
$= \sqrt{(2)^2 + (1)^2}$
$= \sqrt{4+1} = \sqrt{5}$

$d(AR) = \sqrt{(-3-(-1))^2 + (0-1)^2}$
$= \sqrt{(-2)^2 + (-1)^2}$
$= \sqrt{4+1} = \sqrt{5}$

$\therefore AS = AT = AR$
$\therefore$ all are radii
$\therefore$ S, T and R are points on circ. of circle A.

(8) Calculate the value(s) of y for which PQ = QR if P(-2 ; 5) , Q(1 ; 6) and R(0 ; y).

$PQ^2 = QR^2$ if $PQ = QR$

$\therefore \left(\sqrt{(-2-1)^2 + (5-6)^2}\right)^2 = \left(\sqrt{(1-0)^2 + (6-y)^2}\right)^2$

$(-3)^2 + (-1)^2 = (1)^2 + (6-y)^2$

$9 + 1 = 1 + (6-y)^2$

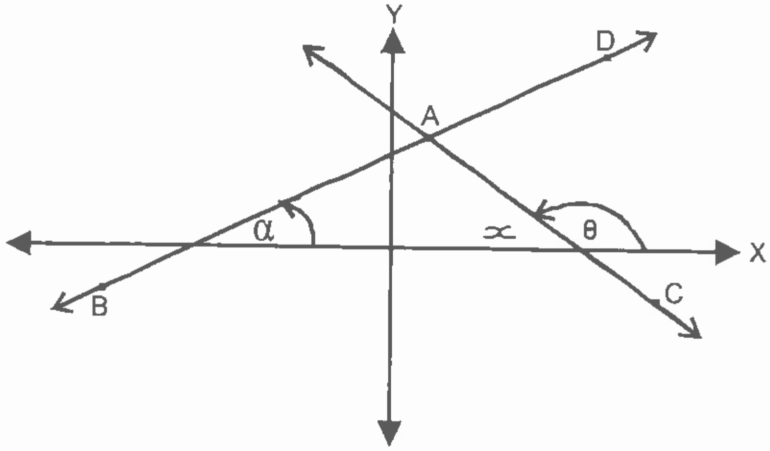
$9 = (6-y)^2$

$\pm 3 = 6 - y$

$3 = 6 - y$ of $-3 = 6 - y$

$y = 3$ $y = 9$

☺ **Calculate the size of $D\hat{A}C$, correct to one decimal, with A(2 ; 5), B(-6 ; -1) and C(7 ; -2):**



$m_{AC} = \frac{y_C - y_A}{x_C - x_A}$ $\qquad$ $m_{AB} = \frac{y_B - y_A}{x_B - x_A}$

$= \frac{-2-5}{7-2}$ $\qquad$ $= \frac{-1-5}{-6-2}$

$= \frac{-7}{5}$ $\qquad$ $= \frac{-6}{-8} = \frac{6}{8}$

$\therefore \tan\theta = -\frac{7}{5}$ $\qquad$ $\therefore \tan\alpha = \frac{6}{8}$

$\therefore \theta = 180° - 54,5°$ $\qquad$ $\therefore \alpha = 36,9°$

$\theta = 125,5°$

$\therefore x = 54,5°$ [∠s on straight line]

$\therefore D\hat{A}C = \alpha + x$

$= 36,9° + 54,5°$

$D\hat{A}C = 91,4°$

C1.3 <u>Mid-point of a line segment</u>:

E.g.5 Calculate the mid-point of line segment PQ with P(-4 ; 5) and Q(2 ; -1).

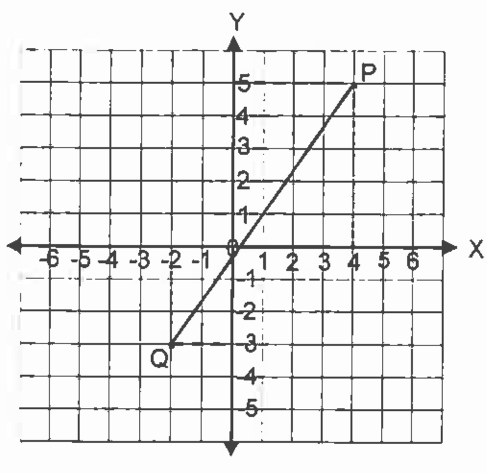
****

The mid-point of PQ, M, will be precisely halfway between P and Q. The x-coordinate of M will be precisely in the middle of the x-coordinates of P and Q and the y-coordinates of M will be precisely in the middle of the y-coordinates of P and Q.

$\therefore M_x = \frac{-2+4}{2} = \frac{2}{2} = \underline{1}$

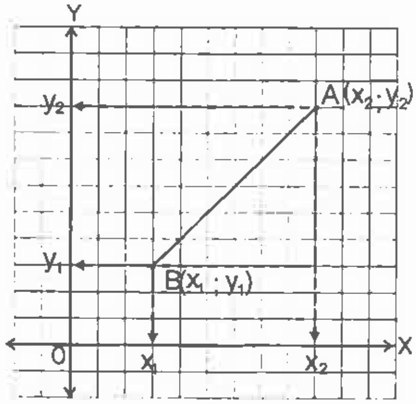
and $M_y = \frac{-3+5}{2} = \frac{2}{2} = \underline{1}$

$\therefore \underline{M = (1 ; 1)}$



Deduction of a formula for the mid-point of any line section between two coordinates:

The mid-point M of line AB lies exactly halfway between A and B.
∴ M's x-coordinate lies exactly halfway between the x-coordinates of A and B and M's y-coordinate lies exactly halfway between A and B's y-coordinates.



$$\therefore x_M = \frac{x_1 + x_2}{2} \quad \text{and} \quad y_M = \frac{y_1 + y_2}{2}$$

$$\therefore M = \left(\frac{x_1 + x_2}{2} ; \frac{y_1 + y_2}{2}\right)$$

E.g.6 Calculate the mid-point between R(-3 ; 2) and T(-4 ; 8).

****

x_1 y_1 x_2 y_2
R(-3 ; 2) and T(-4 ; 8).

$$\therefore M = \left(\frac{x_1 + x_2}{2} ; \frac{y_1 + y_2}{2}\right) = \left(\frac{-3 + (-4)}{2} ; \frac{2 + 8}{2}\right) = \left(\frac{-3 - 4}{2} ; \frac{10}{2}\right)$$

$$\therefore M = \left(\frac{-7}{2} ; 5\right) \quad \text{or} \quad \left(-3\tfrac{1}{2} ; 5\right)$$

Exercise 3: Date:________________

(1) Calculate the mid-point of each of the following line segments:

(a) A(-2 ; 4) and B(-6 ; 4)

$M = \left(\frac{x_1 + x_2}{2} ; \frac{y_1 + y_2}{2}\right)$
$= \left(\frac{-2 - 6}{2} ; \frac{4 + 4}{2}\right)$
$= \left(-\frac{8}{2} ; \frac{8}{2}\right)$
$= (-4 ; 4)$

(b) C(-2 ; 0) and D(0 ; 2)

$M = \left(\frac{x_1 + x_2}{2} ; \frac{y_1 + y_2}{2}\right)$
$= \left(\frac{-2 + 0}{2} ; \frac{0 + 2}{2}\right)$
$= \left(-\frac{2}{2} ; \frac{2}{2}\right)$
$= (-1 ; 1)$

(c) I(-2 ; -7) and J(2 ; 1)

$M = \left(\frac{x_1 + x_2}{2} ; \frac{y_1 + y_2}{2}\right)$
$= \left(\frac{-2 + 2}{2} ; \frac{-7 + 1}{2}\right)$
$= \left(\frac{0}{2} ; \frac{-6}{2}\right)$
$= (0 ; -3)$

(d) K(5 ; 1) and L(11 ; 1)

$M = \left(\frac{x_1 + x_2}{2} ; \frac{y_1 + y_2}{2}\right)$
$= \left(\frac{5 + 11}{2} ; \frac{1 + 1}{2}\right)$
$= \left(\frac{16}{2} ; \frac{2}{2}\right)$
$= (8 ; 1)$