

Grade 12 – Textbook Answers

(First edition – CAPS)

CONTENTS:

	<u>Page:</u>
A1. Sequences and series	3
A2. Logarithms and function inverses	74
A3. Financial Mathematics	100
B1. Differential calculus	148
B2. Probability	226
C1. Trigonometry	253
C2. Data handling	320
D1. Analytical Geometry	353
D2. Euclidean Geometry	398

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Chapter A2

Logarithms and function inverses

See grade 11 Functions and exponents for revision and background!

A2.1 Logarithms:

A2.1.1 Definition of a logarithm:

Logarithms are the inverses of exponents.

Ex. If $2^5 = 32$ then $\log_2 32 = 5$

\therefore Per definition if $y = \log_a x \Leftrightarrow x = a^y$ with $a > 0 ; a \neq 1 \quad x > 0$

Remember: * $\log_a 1 = 0$ because $a^0 = 1$

* The natural logarithm is $\log x \Leftrightarrow \log_{10} x$

* $\log_a a = 1$ because $a^1 = a$

A2.1.2 Laws of logarithms:

For $a > 0 ; a \neq 1 ; b > 0 ; b \neq 1 ; x > 0$ and $y > 0$

- $\log_a x + \log_a y = \log_a xy$
- $\log_a x - \log_a y = \log_a \frac{x}{y}$
- $n \log_a x = \log_a x^n$
- $\log_a x = \frac{\log_b x}{\log_b a}$

Ex. 1 Simplify: (Without using a calculator.)

(a) $\log_4 2 + \log_4 32$

$= \log_4(2 \times 32)$

$= \log_4(64)$

$= \log_4(4^3)$

$= 3\log_4(4)$

$= 3(1)$

$= 3$

(b) $\log 200 - \log 2$

$= \log(200 \div 2)$

$= \log 100$

$= \log_{10} 10^2$

$= 2\log_{10} 10$

$= 2(1)$

$= 2$

(c) $\log_3 36 \times \log_6 9$

$$\begin{aligned}
 &= \frac{\log 36}{\log 3} \times \frac{\log 9}{\log 6} \\
 &= \frac{\log 6^2}{\log 3} \times \frac{\log 3^2}{\log 6} \\
 &= \frac{2 \log 6}{\log 3} \times \frac{2 \log 3}{\log 6} \\
 &= \frac{2 \log 6}{\log 3} \times \frac{2 \log 3}{\log 6} \\
 &= 2 \times 2 \\
 &= 4
 \end{aligned}$$

(d) $\log_4 16 + \log_3 \frac{1}{3} - \log_7 1$

$$\begin{aligned}
 &= \log_4 4^2 + \log_3 3^{-1} - 0 \\
 &= 2 \log_4 4 + (-1) \log_3 3 \\
 &= 2(1) - 1(1) \\
 &= 2 - 1 \\
 &= 1
 \end{aligned}$$

Ex. 2 If $\log 3 = 0,477$ and $\log 5 = 0,699$, calculate:
(Without using a calculator.)

(a) $\log 45$

$$\begin{aligned}
 &= \log(9 \times 5) \\
 &= \log(3^2 \times 5) \\
 &= \log 3^2 + \log 5 \\
 &= 2 \log 3 + \log 5 \\
 &= 2 \times 0,477 + 0,699 \\
 &= 0,954 + 0,699 \\
 &= 1,653
 \end{aligned}$$

(b) $\log 30$

$$\begin{aligned}
 &= \log(3 \times 10) \\
 &= \log 3 + \log 10 \\
 &= \log 3 + \log 10 \\
 &= 0,477 + 1 \\
 &= 1,477
 \end{aligned}$$

Ex. 3 Solve for x : (Without using a calculator.)

(a) $\log x + \log(x + 3) = 1$

$$\begin{aligned}
 \therefore \log_{10} x(x + 3) &= 1 \\
 \therefore 10^1 &= x^2 + 3x \\
 \therefore 0 &= x^2 + 3x - 10 \\
 \therefore 0 &= (x + 5)(x - 2) \\
 \therefore x &= -5 \text{ or } x = 2 \\
 \text{but } x &\neq -5, \text{ because } x > 0
 \end{aligned}$$

(b) $\log_3(x + 4) - \log_3 x = \log_3 5$

$$\begin{aligned}
 \therefore \log_3 \frac{(x+4)}{x} &= \log_3 5 \\
 \therefore \log_3 \frac{(x+4)}{x} &= \log_3 5 \\
 \therefore \frac{(x+4)}{x} &= 5 \quad [\text{Per definition}] \\
 \therefore x + 4 &= 5x \\
 \therefore x - 5x &= -4 \\
 \therefore -4x &= -4 \\
 \therefore x &= 1
 \end{aligned}$$

Ex. 4 Solve for x : (Use a calculator and give your answer correct to 2 decimals.)

(a) $3^x = 7$

$$\begin{aligned}
 \therefore \log_3 7 &= x \\
 \therefore x &= \frac{\log 7}{\log 3} \\
 \therefore x &\approx 1,77
 \end{aligned}$$

(b) $1,3 = 2^{x-3}$

$$\begin{aligned}
 \therefore \log_2 1,3 &= x - 3 \\
 \therefore x - 3 &= \frac{\log 1,3}{\log 2} \\
 \therefore x - 3 &= 0,3785 \dots \\
 \therefore x &\approx 3,38
 \end{aligned}$$

Exercise 1:

(1) Write the following in logarithmic form:

(a) $7^3 = 343$

$$\rightarrow \log_7 343 = 3$$

(b) $x = \left(\frac{1}{2}\right)^2$

$$\rightarrow \log_{\frac{1}{2}} x = 2$$

(c) $y = 2^{x+1}$

$$\rightarrow \log_2 y = x + 1$$

(d) $2^{\log x} = 5$

$$\rightarrow \log_2 5 = \log x$$

(2) Write the following in exponential form:

(a) $\log_2 32 = 5$

$$\rightarrow 2^5 = 32$$

(b) $\log y = k$

$$\rightarrow 10^k = y$$

(c) $m = \log_3 k$

$$\rightarrow 3^m = k$$

(d) $\log_3 \frac{1}{27} = -3$

$$\rightarrow 3^{-3} = \frac{1}{27}$$

(3) Write the following as separate logarithms with base 10 if $\{x; y; t; p\} > 0$:

(a) $\log \frac{xy}{p}$

$$= \log xy - \log p$$

$$= \log x + \log y - \log p$$

(b) $\log_t p^2 t$

$$= 2 \log_t p + \log_t t$$

$$= 2 \frac{\log p}{\log t} + 1$$

(4) Write the following as a single logarithm if $\{x; y; t; p\} > 0$:

(a) $\log t - \log y + 2 \log p$

$$= \log t - \log y + \log p^2$$

$$= \log \frac{tp^2}{y}$$

(b) $\log_2(x-2) - \log_2(x+1) - \log_2 x$

$$= \log_2 \frac{(x-2)}{x(x+1)}$$

(5) Simplify without using a calculator:

(a) $\log 25 + \log 8 - \log 2$

$$= \log(25 \times 8 \div 2)$$

$$= \log 100$$

$$= \log 10^2$$

$$= 2 \log_{10} 10 = 2 \times 1$$

$$= 2$$

(b) $\log_2 16 + 3 \log_3 \left(\frac{1}{9}\right) - \log_{15} 1$

$$= \log_2 2^4 + 3 \log_3 3^{-2} - \log_{15} 15^0$$

$$= 4 \log_2 2 + (3 \times -2) \log_3 3 - 0 \log_{15} 15$$

$$= 4 \times 1 + (-6) \times 1 - 0 \times 1 = 4 - 6 - 0$$

$$= -2$$

(c) $\frac{\log 32 - \log 243}{\log 3 - \log 2}$

$$= \frac{\log 2^5 - \log 3^5}{\log 3 - \log 2}$$

$$= \frac{5 \log 2 - 5 \log 3}{\log 3 - \log 2}$$

$$= \frac{5(\log 2 - \log 3)}{-1(\log 2 - \log 3)}$$

$$= \frac{5(\log 2 - \log 3)}{-1(\log 2 - \log 3)}$$

$$= -5$$

$$\begin{aligned}
 \text{(d)} \quad & \frac{\log_5 27 + \log_5 9}{\log_5 \sqrt{3}} \\
 &= \frac{\log_5 27 \times 9}{\log_5 3^{\frac{1}{2}}} = \frac{\log_5 243}{\frac{1}{2} \log_5 3} \quad \text{or} \quad = \frac{\log_5 3^5}{\frac{1}{2} \log_5 3} \\
 &= 2 \times \log_3 243 \quad = \frac{5 \log_5 3}{\frac{1}{2} \log_5 3} \\
 &= 2 \times \log_3 3^5 \quad = \frac{5 \log_5 3}{\frac{1}{2} \log_5 3} \\
 &= 2 \times 5 \log_3 3 = 2 \times 5 \quad = 5 \div \frac{1}{2} \\
 &= 10 \quad = 10
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad & \log 8\,000 - \log 8 \\
 &= \log \frac{8\,000}{8} \\
 &= \log 1\,000 \\
 &= \log 10^3 \\
 &= 3 \log 10 \\
 &= 3 \times 1 \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad & \frac{1}{2} \log_4 16 + \log_{0,2} 0,04 - \log_3 \sqrt{27} - \log 25 \times \log_5 1 \\
 &= \frac{1}{2} \log_4 4^2 + \log_{0,2} (0,2)^2 - \log_3 \sqrt{3^3} - \log 25 \times 0 \\
 &= \frac{1}{2} \log_4 4^2 + \log_{0,2} (0,2)^2 - \log_3 \sqrt{3^3} - \log 25 \times 0 \\
 &= \frac{1}{2} \times 2 \times \log_4 4 + 2 \times \log_{0,2} 0,2 - \log_3 3^{\frac{3}{2}} - 0 \\
 &= \frac{1}{2} \times 2 \times 1 + 2 \times 1 - \frac{3}{2} \times \log_3 3 \\
 &= 1 + 2 - \frac{3}{2} \times 1 \\
 &= 3 - \frac{3}{2} \\
 &= \frac{3}{2}
 \end{aligned}$$

(6) Solve for x : [Where necessary, round off correct to 2 decimals.]

(a) $\log_4 2x = 3$

$$\therefore 4^3 = 2x$$

$$\therefore 64 = 2x$$

$$\therefore x = 32$$

(b) $\log_3(x + 2) + \log_3 x = 1$

$$\therefore \log_3 x(x + 2) = 1 \quad \text{or} \quad \log_3 x(x + 2) = \log_3 3$$

$$\therefore 3^1 = x^2 + 2x \quad \leftarrow \quad \text{EB} \Leftrightarrow \text{EE}$$

$$\therefore 0 = x^2 + 2x - 3$$

$$\therefore 0 = (x + 3)(x - 1)$$

$$\therefore x = -3 \quad \text{or} \quad x = 1$$

N/A \rightarrow **Ex. $\log_3 -3$ is not admissible by definition**

(c) $\log_2(2x + 12) - 2 = \log_2 x$

$$\therefore \log_2(2x + 12) - 2 \log_2 2 = \log_2 x$$

$$\therefore \log_2(2x + 12) - \log_2 2^2 = \log_2 x$$

$$\therefore \log_2 \frac{(2x+12)}{4} = \log_2 x$$

$$\therefore \frac{(2x+12)}{4} = x \quad \leftarrow \quad \text{EB} \Leftrightarrow \text{EE}$$

$$\therefore 2x + 12 = 4x$$

$$\therefore 12 = 4x - 2x$$

$$\therefore 2x = 12$$

$$\therefore x = 6$$

$$(d) \quad 7^{3x} = 14$$

$$\therefore \log_7 14 = 3x$$

$$\therefore \frac{\log 14}{\log 7} = 3x$$

$$\therefore 3x = 1,356 \dots \quad \rightarrow \quad \text{Use a calculator}$$

$$\therefore 3x = 1,356 \dots$$

$$\therefore x \approx 0,45$$

(7) Write the following in terms of m and/or n if $\log 6 = m$ and $\log 3 = n$:

$$(a) \quad \log 18$$

$$= \log 3 \times 6$$

$$= \log 3 + \log 6$$

$$= m + n$$

$$(b) \quad \log_{27} 36$$

$$= \frac{\log 36}{\log 27} = \frac{\log 6^2}{\log 3^3}$$

$$= \frac{2 \log 6}{3 \log 3}$$

$$= \frac{2m}{3n}$$

$$(c) \quad \log 300$$

$$= \log 3 \times 100$$

$$= \log 3 + \log 100$$

$$= \log 3 + \log 10^2 = \log 3 + 2 \log 10$$

$$= n + 2$$

$$(d) \quad \log 20$$

$$= \log \frac{60}{3}$$

$$= \log 6 \times 10 - \log 3$$

$$= \log 6 + \log 10 - \log 3$$

$$= m + 1 - n$$

A2.2 Inverses:

The rule for the reflection in the line $x = y$ is: $(x ; y) \Leftrightarrow (y ; x)$

This reflection in the line $y = x$ is referred to as the inverse \Leftrightarrow it means that the x and y swap places!

The inverse of $f(x)$ is written as $f^{-1}(x)$.

Ex. 5 Determine $f^{-1}(x)$ in each of the following in the form $f^{-1}(x) = \dots$:

(a) $f(x) = 5x^2$

\therefore For f : $y = 5x^2$

\therefore For f^{-1} : $x = 5y^2$

$\therefore \frac{x}{5} = y^2$

$\therefore y = \pm \sqrt{\frac{x}{5}}$

$\therefore f^{-1}(x) = \pm \sqrt{\frac{x}{5}}$

(b) $f: x \rightarrow \frac{3}{x+2}$

\therefore For f : $y = \frac{3}{x+2}$

\therefore For f^{-1} : $x = \frac{3}{y+2}$

$\therefore y + 2 = \frac{3}{x}$

$\therefore y = \frac{3}{x} - 2$

$\therefore f^{-1}(x) = \frac{3}{x} - 2$

Exercise 2:

(1) Determine $f^{-1}(x)$ in each of the following and write it in the form $f^{-1}(x) = \dots$

(a) $f(x) = 3x - 4 \rightarrow y = 3x - 4$

f^{-1} : $x = 3y - 4$

$\therefore 3y = x + 4$

$\therefore y = \frac{x+4}{3}$

$\therefore f^{-1}(x) = \frac{x+4}{3}$

(b) $f(x) = 5^x \rightarrow y = 5^x$

f^{-1} : $x = 5^y$

$\therefore y = \log_5 x$

$\therefore f^{-1}(x) = \log_5 x$

(c) $f(x) = -2x^2 \rightarrow y = -2x^2$

f^{-1} : $x = -2y^2$

$\therefore y^2 = \frac{x}{-2}$

$\therefore y = \pm \sqrt{-\frac{x}{2}}$

$\therefore f^{-1}(x) = y = \pm \sqrt{-\frac{x}{2}}$

(d) $f(x) = \log_{0,5} x \rightarrow y = \log_{0,5} x$

f^{-1} : $x = \log_{0,5} y$

$\therefore y = 0,5^x$

$\therefore f^{-1}(x) = 0,5^x$

(2) Determine $g^{-1}(x)$ in each of the following and write it in the form $g^{-1}: x \rightarrow \dots\dots$

(a) $g : x \rightarrow \frac{x}{4} \rightarrow y = \frac{x}{4}$

$g^{-1}: x = \frac{y}{4}$

$\therefore y = 4x$

$\therefore g^{-1}: x \rightarrow 4x$

(b) $g : x \rightarrow \log_3 x \rightarrow y = \log_3 x$

$g^{-1}: x = \log_3 y$

$\therefore y = 3^x$

$\therefore g^{-1}: x \rightarrow 3^x$

(c) $g : x \rightarrow 3^{x+1} \rightarrow y = 3^{x+1}$

$g^{-1}: x = 3^{y+1}$

$\therefore y + 1 = \log_3 x$

$\therefore y = \log_3 x - 1$

$\therefore g^{-1}: x \rightarrow \log_3 x - 1$

(d) $g : x \rightarrow -0,5x \rightarrow y = -0,5x$

$g^{-1}: x = -0,5y$

$\therefore y = \frac{x}{-0,5}$

$\therefore g^{-1}: x \rightarrow -\frac{x}{0,5}$

(3) Determine h in each of the following and write it in the form $h(x) = \dots\dots$

(a) $h^{-1}(x) = \log_7 x \rightarrow y = \log_7 x$

$h: x = \log_7 y$

$\therefore y = 7^x$

$\therefore h(x) = 7^x$

(b) $h^{-1}(x) = \frac{x-2}{3} \rightarrow y = \frac{x-2}{3}$

$h: x = \frac{y-2}{3}$

$\therefore 3x = y - 2$

$\therefore y = 3x + 2$

$\therefore h(x) = 3x + 2$

(c) $h^{-1}(x) = \frac{1}{4}x^2 \rightarrow y = \frac{1}{4}x^2$

$h: x = \frac{1}{4}y^2$

$\therefore 4x = y^2$

$\therefore y = \pm\sqrt{4x}$

$\therefore h(x) = \pm\sqrt{4x}$

(d) $h^{-1}(x) = \log x \rightarrow y = \log x$

$h: x = \log y$

$\therefore y = 10^x$

$\therefore h(x) = 10^x$

(4) Consider the following: $p(x) = \{(1; 7); (2; 8); (3; 9); (4; 10)\}$

(a) Is p a function? Motivate your answer.

Yes, p is a function, because none of the x -coordinates are repeated.

(b) Write down the range of $p^{-1}(x)$.

$$\mathbf{R_{p^{-1}} = \{1; 2; 3; 4\}} \quad \rightarrow \quad \mathbf{R_{p^{-1}} = D_p}$$

(5) Explain the difference between $f^{-1}(x)$ and $(f(x))^{-1}$.

$f^{-1}(x)$ is the inverse of a function and $(f(x))^{-1}$ is the reciprocal of the function.

$$\therefore \text{If } f(x) = 2x, \text{ then } f^{-1}(x) = \frac{x}{2} \text{ and } (f(x))^{-1} = \frac{1}{2x}.$$

$$\rightarrow y = 2x \rightarrow \text{For } f^{-1}: x = 2y$$

A2.3 Graphs of inverses:

A2.3.1 Graphs of inverses of the straight line:

See grade 11 Linear Functions for revision and background!

If the function $f(x) = mx + c$ is given, the inverse will be obtained as follows:

$$f(x) = mx + c \Leftrightarrow y = mx + c$$

\therefore For the inverse the x and y swap places: $x = my + c$

$$\Rightarrow y = \frac{x - c}{m} \quad \text{[Make } y \text{ the subject!]}$$

$$\therefore f^{-1}(x) = \frac{x - c}{m} \Rightarrow \text{Inverse function}$$

Ex. 6 Given: $g(x) = 2x - 4$

(a) Determine $g^{-1}(x) = \dots$

(b) Sketch $g(x)$ and $g^{-1}(x)$ on the same system of axes.

$$(a) \quad g(x): y = 2x - 4 \Leftrightarrow \therefore g^{-1}(x): x = 2y - 4$$

$$\therefore 2y = x + 4$$

$$\therefore y = \frac{1}{2}x + 2$$

$$\therefore g^{-1}(x) = \frac{1}{2}x + 2$$

(b) For $g(x)$: x -intercept ($y = 0$) y -intercept ($x = 0$)

$$2x - 4 = 0$$

$$\therefore x = 2$$

$$\therefore (2 ; 0) \quad \text{and}$$

$$y = 2(0) - 4$$

$$\therefore y = -4$$

$$(0 ; -4)$$

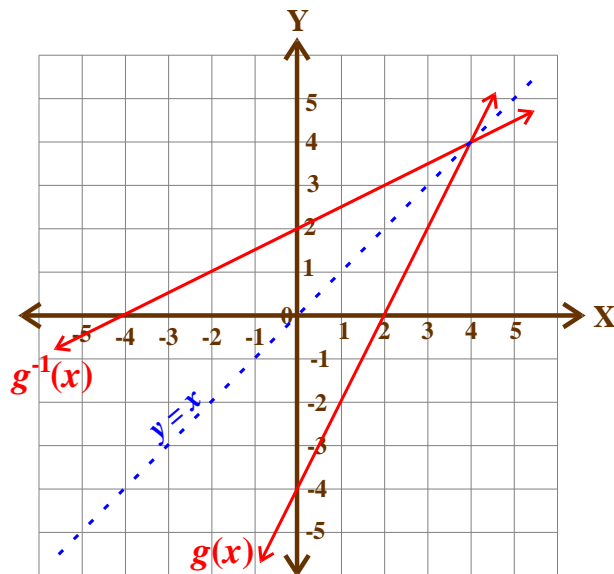
For $g^{-1}(x)$: x and y of $g(x)$ swap place:

$$y\text{-intercept } (x = 0)$$

$$\therefore (0 ; 2)$$

$$x\text{-intercept } (y = 0)$$

$$(-4 ; 0)$$



A2.3.2 Graphs of inverses of the parabola:

See grade 11 Quadratic Functions for revision and background!

Ex. 7 Given: $g(x) = 2x^2$ with $x \geq 0$

- (a) Determine $g^{-1}(x) = \dots$
 (b) Sketch $g(x)$ and $g^{-1}(x)$ on the same system of axes.
 (c) Write down the domain of $g^{-1}(x)$.

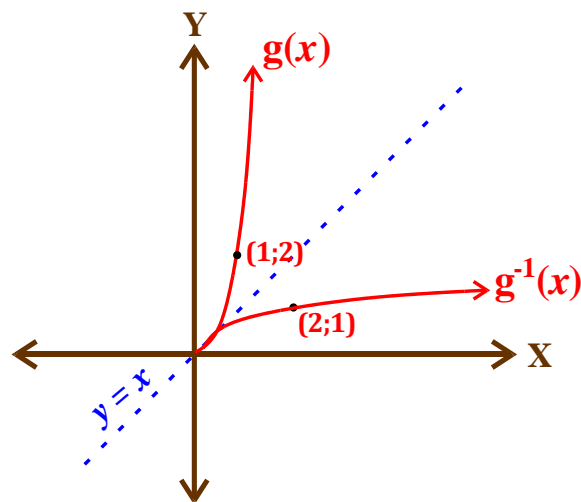
$$\begin{aligned} \text{(a) } g(x): y = 2x^2 \text{ with } x \geq 0 &\Leftrightarrow \therefore g^{-1}(x): x = 2y^2 \text{ with } y \geq 0 \\ &\therefore y^2 = \frac{x}{2} \\ &\therefore y = \pm \sqrt{\frac{x}{2}} \\ &\text{but } y \geq 0 \\ &\therefore g^{-1}(x) = +\sqrt{\frac{x}{2}} \end{aligned}$$

(b) For $g(x)$: Use a table, because the x -and- y intercepts and the turning point is $(0 ; 0)$.

x	0	1	2	$x \geq 0$
y	0	2	8	

For $g^{-1}(x)$: x and y of $g(x)$ swap place:

x	0	2	8
y	0	1	2



(c) $D_{g^{-1}}: x \geq 0$

A2.3.3 Graphs of inverses of the exponential function:

See grade 11 Exponential Functions for revision and background!

If the function $f(x) = a^x$ is given, the inverse will be obtained as follows:

$$f(x) = a^x \Leftrightarrow y = a^x$$

\therefore For the inverse, the x and y swap places: $x = a^y$

$$\Rightarrow y = \log_a x \quad \text{[Make } y \text{ the subject!]}$$

\therefore The inverse of an exponential function is a logarithmic function.

Ex. 8 Given: $g(x) = 2^x$

(a) Determine $g^{-1}(x) = \dots$

(b) Sketch $g(x)$ and $g^{-1}(x)$ on the same system of axes.

(c) Write down the equation of the asymptote of $g^{-1}(x)$.

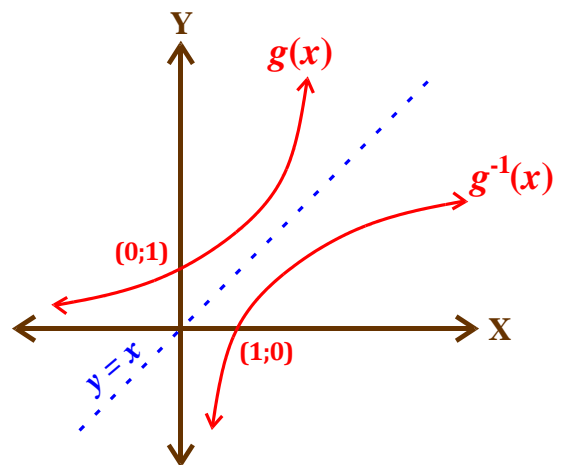
$$\begin{aligned} \text{(a) } g(x): y = 2^x & \Leftrightarrow \therefore g^{-1}(x): x = 2^y \\ & \therefore y = \log_2 x \\ & \therefore g^{-1}(x) = \log_2 x \end{aligned}$$

(b) For $g(x)$:

x	-1	0	1
y	$\frac{1}{2}$	1	2

For $g^{-1}(x)$: x and y of $g(x)$ swap places:

x	$\frac{1}{2}$	1	2
y	-1	0	1



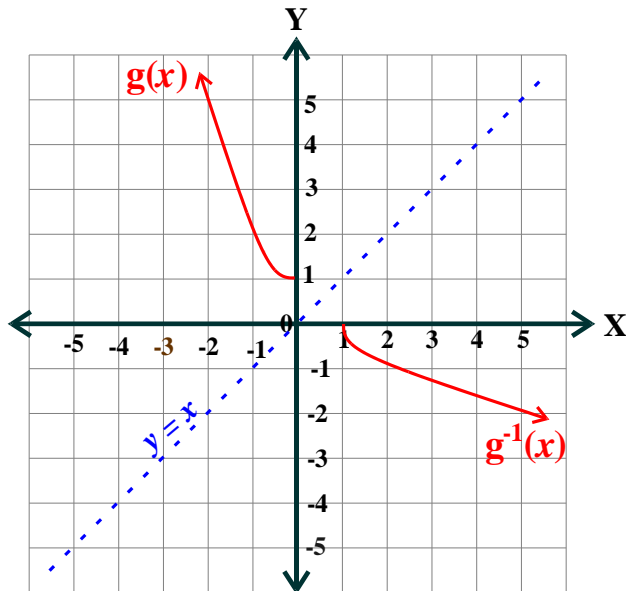
(c) Asymptote of $g^{-1}(x)$:

$$x = 0$$

Exercise 3:

- (1) (a) Sketch:
- $g(x) = x^2 + 1$
- for
- $x \leq 0$
- .

x	0	-1	-2	$x \leq 0$
y	1	2	5	



- (b) Determine
- g^{-1}
- and write it in the form
- $g^{-1}(x) = \dots\dots$
- $g(x) = x^2 + 1 = y$

$$g^{-1}: \quad y^2 + 1 = x \quad \text{for} \quad y \leq 0$$

$$\therefore y^2 = x - 1$$

$$\therefore y = \pm\sqrt{x-1} \rightarrow g^{-1}(x) = -\sqrt{x-1}$$

- (c) Sketch
- $g^{-1}(x)$
- on the same system of axes as
- $g(x)$
- .

x	1	2	5	$y \leq 0$
y	0	-1	-2	

- (d) Write down the range of
- $g^{-1}(x)$
- .

$$R_{g^{-1}}: y \leq 0$$

- (2) Given:
- $h(x) = 2^{-x} \rightarrow h(x) = 2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x = y$

- (a) Determine
- h^{-1}
- and write it in the form
- $h^{-1}(x) = \dots\dots$

$$h^{-1}: \quad \left(\frac{1}{2}\right)^y = x$$

$$\therefore y = \log_{\frac{1}{2}} x$$

$$\therefore h^{-1} = \log_{\frac{1}{2}} x$$

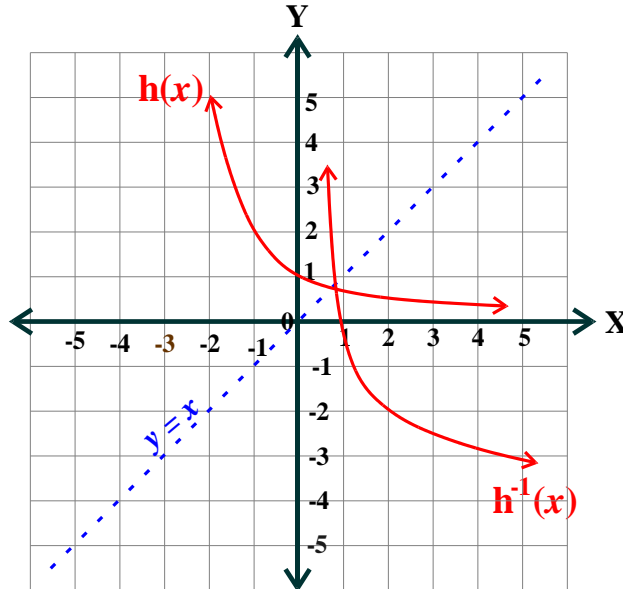
(b) Sketch h and h^{-1} on the same system of axes.

$h:$	x	-1	0	1
	y	2	1	$\frac{1}{2}$

Asymptote: $y = 0$

$h^{-1}:$	x	2	1	$\frac{1}{2}$
	y	-1	0	1

Asymptote: $x = 0$



(c) Write down the domain of $h^{-1}(x)$. **$D_{h^{-1}}: x > 0$**

(d) If p is the reflection of h in the y -axis, determine the equation of p and write it in the form $p(x) = \dots\dots$

$$p(x) = \left(\frac{1}{2}\right)^{-x}$$

$\therefore p(x) = 2^x$

(e) Determine p^{-1} and write it in the form $p^{-1}(x) = \dots\dots$

$\therefore p^{-1}(x) = \log_2 x$

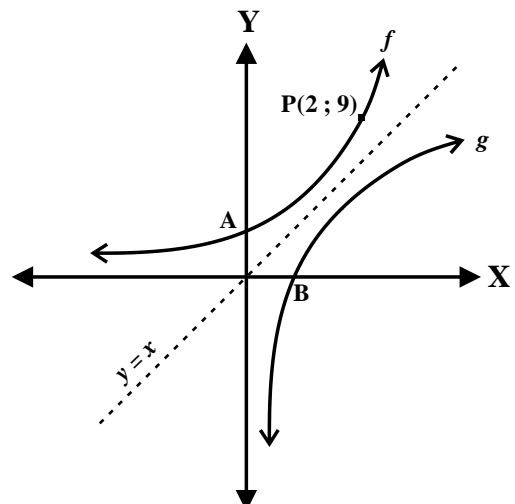
(3) Given: $f(x) = a^x$ and $g(x)$ with $P(2; 9)$.

(a) Determine the value of a .

$$y = f(x) = a^x \quad \text{through} \quad \begin{matrix} x & y \\ P(2; 9) \end{matrix}$$

$$\therefore 9 = a^2$$

$\therefore a = 3$ $\rightarrow a > 0$



(b) Give the coordinates of A.

$A(0; 1)$

- (c) Determine the equation of $g(x)$, if $g(x)$ is the mirror image of $f(x)$ in the line $y = x$.

$g(x)$ is the inverse of $f(x)$ with $f(x) = y = 3^x$

$$\therefore \text{ for } g: \quad x = 3^y$$

$$\therefore y = g(x) = \log_3 x$$

- (d) Give the coordinates of B. **B(1; 0)**

- (e) For which values of x will $g(x)$ be defined?

$$\text{Defined} \rightarrow \text{Domain} \quad \therefore D_g: x > 0$$

- (f) Write down the equation of the asymptote of $g(x)$. **y-axis** \rightarrow **$x = 0$**

- (4) Given: $t(x) = a^x$ and $p(x) = bx^2$

met $A(-2; 4)$.

- (a) Determine the values of a and b .

$$t(x) = y = a^x \quad \text{through} \quad \begin{matrix} x & y \\ A(-2 & 4) \end{matrix}$$

$$\therefore 4 = a^{-2}$$

$$\therefore 4 = \frac{1}{a^2}$$

$$\therefore a^2 = \frac{1}{4}$$

$$\therefore a = \frac{1}{2} \quad \rightarrow \quad a > 0$$

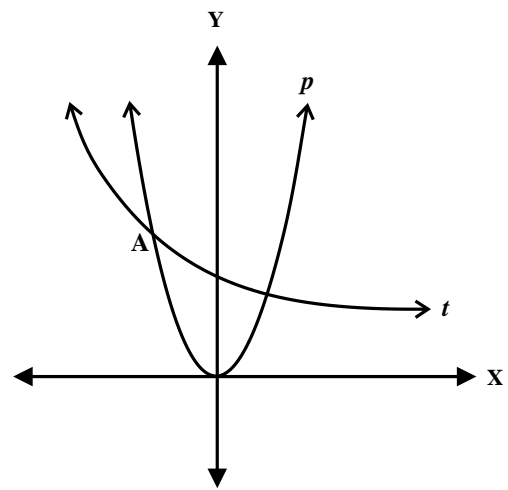
$$p(x) = y = bx^2 \quad \text{through} \quad \begin{matrix} x & y \\ A(-2 & 4) \end{matrix}$$

$$\therefore 4 = b(-2)^2$$

$$\therefore 4 = b(4)$$

$$\therefore 4 = b(4)$$

$$\therefore b = 1$$



- (b) Write down the following: $t^{-1}(x) = \dots\dots$

$$t(x) = y = \left(\frac{1}{2}\right)^x$$

$$\text{For } t^{-1}(x): \quad x = \left(\frac{1}{2}\right)^y$$

$$\therefore y = \log_{\frac{1}{2}} x$$

$$\therefore t^{-1}(x) = \log_{\frac{1}{2}} x$$

- (c) Write down the following: $p^{-1}(x) = \dots\dots$

$$p(x) = 1x^2 \quad \rightarrow \quad y = x^2$$

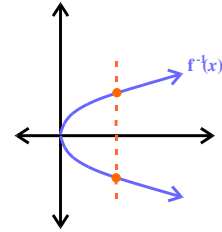
$$\text{For } p^{-1}(x): \quad x = y^2$$

$$\therefore y = \pm\sqrt{x}$$

$$\therefore p^{-1}(x) = \pm\sqrt{x}$$

- (d) Explain why $p^{-1}(x)$ is not a function.

**Because $p^{-1}(x)$ each input does not have a unique output.
See sketch.**

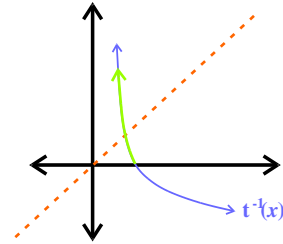


- (e) Determine x for which $t^{-1}(x) \geq 0$.

$$\therefore y \geq 0$$

$$\therefore 0 < x \leq 1$$

See the solution in green.



- (f) Calculate: $t^{-1}(0,25) + p(3)$

$$= \log_{\frac{1}{2}} 0,25 + (3)^2$$

$$= 2 + 9$$

$$= 11$$

- (5) The graph of $f(x) = a^x$ is sketched alongside.
The point $B(3; 8)$ lies on the graph of f .

- (a) Show that $a = 2$.

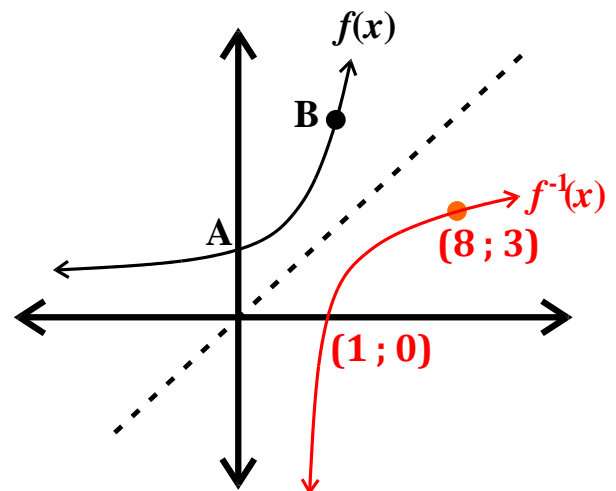
$$f(x) = y = a^x \quad \text{through} \quad \begin{matrix} x & y \\ B(3; 8) \end{matrix}$$

$$\therefore 8 = a^3$$

$$\therefore 2^3 = a^3$$

$$\therefore a = 2$$

$$EB \Leftrightarrow EE$$



- (b) Write down the coordinates of A.

$$\mathbf{A(0; 1)} \quad \text{because} \quad y = 2^0 = 1$$

- (c) Write down the equation of $f^{-1}(x)$

in the form $f^{-1}(x) = \dots$

$$f(x) = y = 2^x$$

$$\text{For } f^{-1}(x): \quad x = 2^y$$

$$\therefore f^{-1}(x) = y = \log_2 x$$

- (d) Sketch the graph of f^{-1} .

Show the x -intercept and ONE other point. See sketch.

- (e) For which values of x will $f^{-1}(x) = f(x)$?

None $\rightarrow f^{-1}(x)$ and $f(x)$ don't intersect.

- (f) Write down the equation of g if g is the reflection of f in the y -axis.

$$g(x) = \left(\frac{1}{2}\right)^x \text{ or } g(x) = 2^{-x}$$

- (g) Write down the equation of h if h is the reflection of f^{-1} in the x -axis.

$$h(x) = -\log_2 x$$

- (h) Are g and h one another's inverse? Motivate your answer.

$$\begin{aligned} \text{Yes, because } g(x) = y = 2^{-x} &\rightarrow g^{-1}: x = 2^{-y} \\ &\therefore -y = \log_2 x \\ &\therefore y = -\log_2 x = h(x) \end{aligned}$$

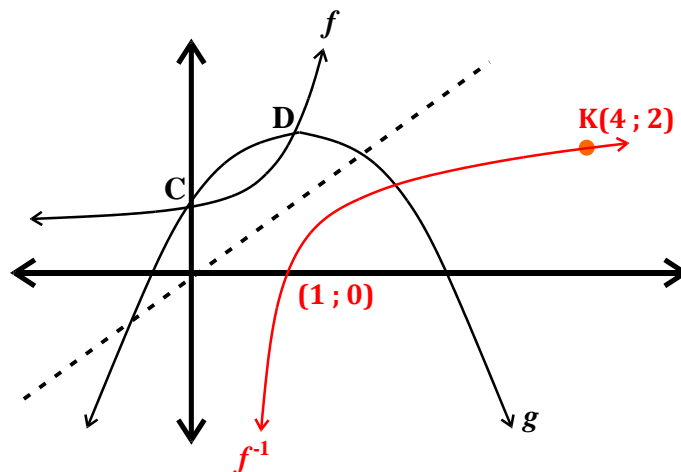
- (i) For which values of x will $f^{-1}(x) \geq 0$? $\rightarrow y \geq 0$ for f^{-1}

$$\therefore x \geq 1$$

- (j) Calculate: $f^{-1}(2) + f(-2)$

$$\begin{aligned} &= \log_2 2 + 2^{-2} \\ &= 1 + \frac{1}{2^2} \\ &= 1 + \frac{1}{4} = 1\frac{1}{4} \end{aligned}$$

- (6) On the right is the graphs of $f(x) = 2^x$ and $g(x) = -(x-1)^2 + b$, with b as a constant value. The graphs of f and g intersect on the y -axis at C . D is the turning point of g .



- (a) Show that $b = 2$.

C is the y -intercept of both f and g .

For f : $y = 2^x$ with $x = 0 \rightarrow$ for y -intercept

$$\therefore y = 2^0 = 1 \rightarrow C(0; 1)$$

For g : $y = -(x-1)^2 + b$ through $\begin{matrix} x & y \\ C(0 & 1) \end{matrix}$

$$\therefore 1 = -(0-1)^2 + b$$

$$\therefore 1 = -(-1)^2 + b$$

$$\therefore 1 = -1 + b$$

$$\therefore b = 2$$

(b) Write down the coordinates of the turning point of g . **TP: (1; 2)**

(c) Write down the equation of $f^{-1}(x)$ in the form $y = \dots\dots$

$$f^{-1}(x) = y = \log_2 x$$

(d) Sketch the graph of f^{-1} on the same graph as given above.

Show on your graph the x -intercept and the coordinates of one other point.

See graph!

Other point: If $x = 4 \rightarrow y = \log_2 4 = 2$

$$\therefore \mathbf{K(4; 2)}$$

(e) Write down the equation of h if $h(x) = g(x + 1) - 2$.

$$h(x) = [-((x + 1) - 1)^2 + 2] - 2$$

$$h(x) = [-(x + 1 - 1)^2 + 2] - 2$$

$$h(x) = [-(x)^2 + 2] - 2$$

$$h(x) = -x^2 + 2 - 2$$

$$h(x) = -x^2$$

(f) How can the domain of h be restricted so that h^{-1} will be a function?

The domain of h should be limited as below for h^{-1} to be a function:

$$x \leq 0 \quad \text{or} \quad x \geq 0$$

\rightarrow The range of h^{-1} will be as follow:

$$y \leq 0 \quad \text{or} \quad y \geq 0$$

(g) Determine the maximum value of $2^2 - (x - 1)^2$.

The maximum value of $2^2 - (x - 1)^2$ will be at the maximum value of $[2 - (x - 1)^2]$

$$\therefore 2 - (x - 1)^2 = - (x - 1)^2 + 2$$

\rightarrow maximum value is 2 \Rightarrow read from the turning point (1 ; 2)

$$\therefore \text{maximum value of } 2^2 - (x - 1)^2 = 2^2$$

$$\therefore \text{maximum value of } 2^2 - (x - 1)^2 = 4$$

REVISION FROM PAST PAPERS:Exercise A:

Consider the function $f(x) = y = \left(\frac{1}{3}\right)^x \rightarrow y = 3^{-x}$

- (1) Is f an increasing or decreasing function? Give a reason for your answer. (2)

Decreasing function ✓ **Reason: $0 < a < 1$** ✓ or **As x increases, $f(x)$ will decrease.**

- (2) Calculate $f^{-1}(x)$ in the form $y = \dots\dots\dots$ (2)

$$f^{-1}(x): \quad x = \left(\frac{1}{3}\right)^y \quad \checkmark \quad \text{or} \quad x = 3^{-y}$$

$$\therefore y = \log_{\frac{1}{3}} x \quad \checkmark \quad \therefore y = -\log_3 x$$

- (3) Write down the equation of the asymptote of $f(x) - 5$. (1)

$$y = -5 \quad \checkmark$$

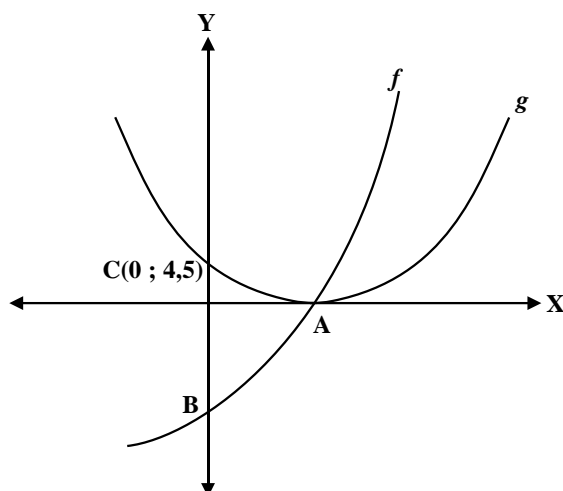
- (4) Describe the transformation of f to g if $g(x) = \log_3 x$. (2)

A reflection in the line $y = x$ followed by a reflection in the x -axis. ✓✓

or **A reflection in the y -axis followed by a reflection in the line $y = x$.**

Exercise B:

The graphs of $f(x) = 2^x - 8$ and $g(x) = ax^2 + bx + c$ sketched below. B and C(0 ; 4,5) are the y-intercepts of the graphs of f and g respectively. The two graphs intersect at A, which is the turning point of the graph of g and the x-intercept of the graphs of f and g .



- (1) Determine the coordinates of A and B. (4)

A: x-intercept of $g \rightarrow y = 0$

$$\rightarrow 0 = 2^x - 8$$

$$\rightarrow 8 = 2^3 = 2^x$$

$$\rightarrow \therefore x = 3$$

$$\therefore \mathbf{A(3; 0)} \quad \checkmark$$

B: y-intercept of $g \rightarrow x = 0$

$$\rightarrow y = 2^0 - 8$$

$$\rightarrow y = 1 - 8$$

$$\rightarrow y = -7$$

$$\therefore \mathbf{B(0; -7)} \quad \checkmark$$

- (2) Write down the equation of the asymptote of graph f . (1)

$$\mathbf{y = -8} \quad \checkmark$$

- (3) Determine the equation of h if $h(x) = f(2x) + 8$. (2)

$$\mathbf{h(x) = f(2x) + 8 = 2^{2x} - 8 + 8 \rightarrow h(x) = 2^{2x} = 4^x} \quad \checkmark \checkmark$$

- (4) Determine the equation of h^{-1} in the form $y = \dots$ (2)

$$\mathbf{h^{-1}: \quad x = 4^y \quad \rightarrow \quad y = \log_4 x} \quad \checkmark \checkmark$$

- (5) Write down the equation of p , if p is the reflection of h^{-1} in the x -axis. (1)

$$\mathbf{p(x) = -\log_4 x \quad \text{or} \quad p(x) = \log_{\frac{1}{4}} x} \quad \checkmark$$

- (6) Calculate $\sum_{k=0}^3 g(k) - \sum_{k=4}^5 g(k)$. Show ALL calculations. (4)

$$= [g(0) + g(1) + g(2) + g(3)] - [g(4) + g(5)] \quad \checkmark$$

$$= g(0) + g(1) + g(2) + g(3) - g(4) - g(5) \quad \text{with } x = 3 \text{ the line of symmetry}$$

$$= g(0) + g(3) \quad \checkmark \quad \text{because } g(1) = g(5) \quad \text{and} \quad g(2) = g(4)$$

$$= 4,5 - 0 \quad \checkmark \quad \rightarrow \quad \text{Given } C(0 ; 4,5) \quad \text{and calculated in (a) } A(3 ; 0)$$

$$= 4,5 \quad \checkmark$$

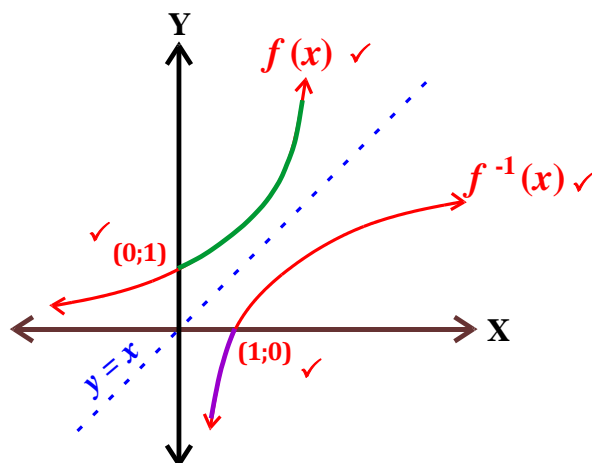
Exercise C:

Given: $f(x) = 3^x$

- (1) Determine an equation for f^{-1} in the form $f^{-1}(x) = \dots$ (1)

$$f^{-1}(x) = \log_3 x \quad \checkmark$$

- (2) Sketch the graphs of f and f^{-1} , clearly showing ALL intercepts with the axes. (4)



- (3) Write down the domain of f^{-1} . (2)

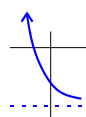
$$D_{f^{-1}}: x > 0 \quad \text{or} \quad x \in (0 ; \infty) \quad \checkmark \checkmark$$

- (4) For which values of x will $f(x) \cdot f^{-1}(x) \leq 0$? (2)

$$0 < x < 1 \quad \checkmark \checkmark \quad \text{where } f(x) > 0 \quad \text{and} \quad f^{-1}(x) < 0$$

- (5) Write down the range of $h(x) = 3^{-x} - 4$. (2)

$$R_h: y > -4 \quad \text{or} \quad y \in (-4 ; \infty) \quad \checkmark \checkmark$$



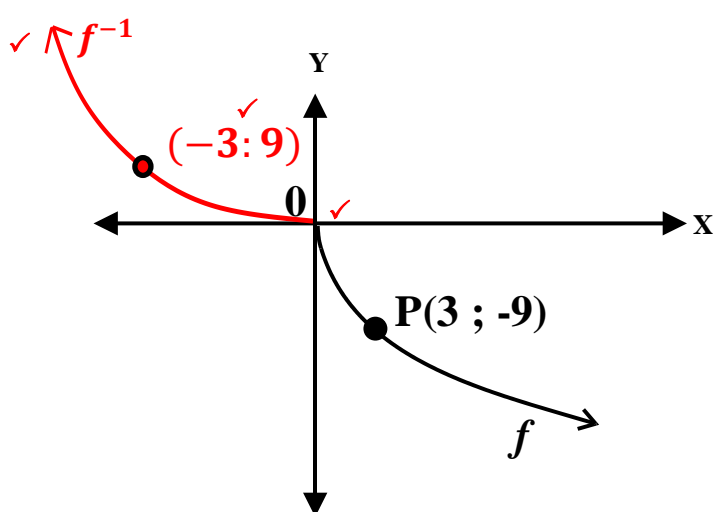
- (6) Write down an equation for g if the graph of g is the image of the graph of f after f has been translated two units to the right and reflected about the x -axis. (2)

$$g(x) = -3^{x-2} \quad \checkmark\checkmark$$

Exercise D:

The graph of $f(x) = -\sqrt{27x}$ for $x \geq 0$ is sketched below.

The point $P(3; -9)$ lies on the graph of f .



- (1) Use the graph to determine the values of x for which $f(x) \geq -9$. (2)

$$0 \leq x \leq 3 \quad \checkmark\checkmark$$

For $0 < x < 3$ only 1 mark

- (2) Write down the equation of f^{-1} in the form $y = \dots$. Indicate ALL restrictions. (3)

$$f^{-1}: \quad x = -\sqrt{27y} \quad \rightarrow \quad (x)^2 = (-\sqrt{27y})^2 \quad \checkmark$$

$$\therefore \quad x^2 = 27y$$

$$\therefore \quad y = \frac{x^2}{27} = \frac{1}{27}x^2 \quad \text{for} \quad x \leq 0 \quad \text{or} \quad x \in (-\infty; 0) \quad \checkmark$$

- (3) Sketch f^{-1} , the inverse of f . Indicate the intercept(s) with the axes and the coordinates of ONE other point. (3)

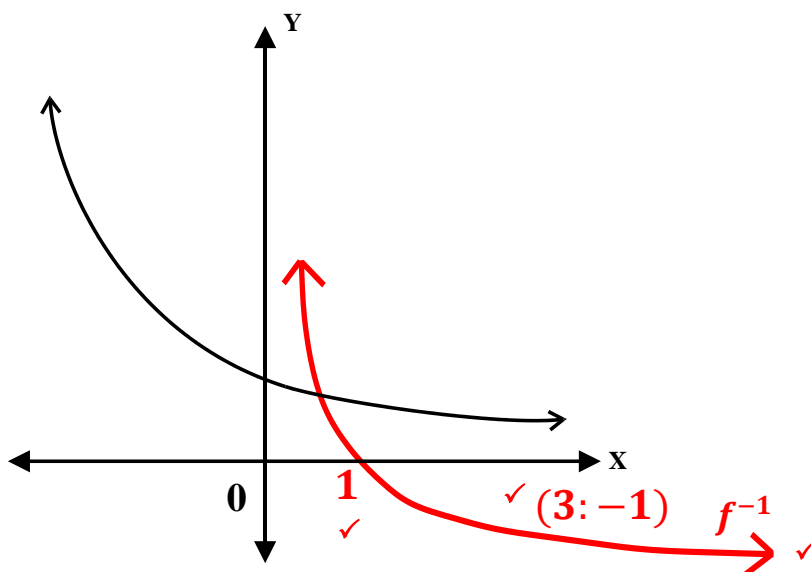
See sketch above!

- (4) Describe the transformation of f to g if $g(x) = \sqrt{27x}$ for $x \geq 0$. (1)

Reflection about the x -axis or $(x; y) \rightarrow (x; -y)$ for $x \geq 0$ \checkmark

Exercise E:

The graph of $f(x) = \left(\frac{1}{3}\right)^x$ is sketched below.



- (1) Write down the domain of f . (1)

\mathbb{R} or $x \in (-\infty; \infty)$ ✓

- (2) Write down the equation of the asymptote of f . (1)

$y = 0$ ✓

- (3) Write down the equation of f^{-1} in the form $y = \dots$ (2)

$f^{-1}: x = \left(\frac{1}{3}\right)^y = 3^{-y}$ ✓ $\rightarrow y = \log_{\frac{1}{3}} x = -\log_3 x$ ✓

- (4) Sketch the graph of f^{-1} . Indicate the x -intercept and the coordinates of ONE other point. See sketch above! (3)

- (5) Write down the equation of the asymptote of $f^{-1}(x+2)$. (2)

$x = -2$ ✓✓ Graph moves two units to the left

- (6) Prove that: $[f(x)]^2 - [f(-x)]^2 = f(2x) - f(-2x)$ for all values of x . (3)

$$\text{LHS} = [f(x)]^2 - [f(-x)]^2 = \left[\left(\frac{1}{3}\right)^x\right]^2 - \left[\left(\frac{1}{3}\right)^{-x}\right]^2 \quad \checkmark$$

$$\therefore \text{LHS} = 3^{-2x} - 3^{2x} \quad \checkmark$$

$$\text{RHS} = f(2x) - f(-2x) = \left(\frac{1}{3}\right)^{2x} - \left(\frac{1}{3}\right)^{-2x} \quad \checkmark$$

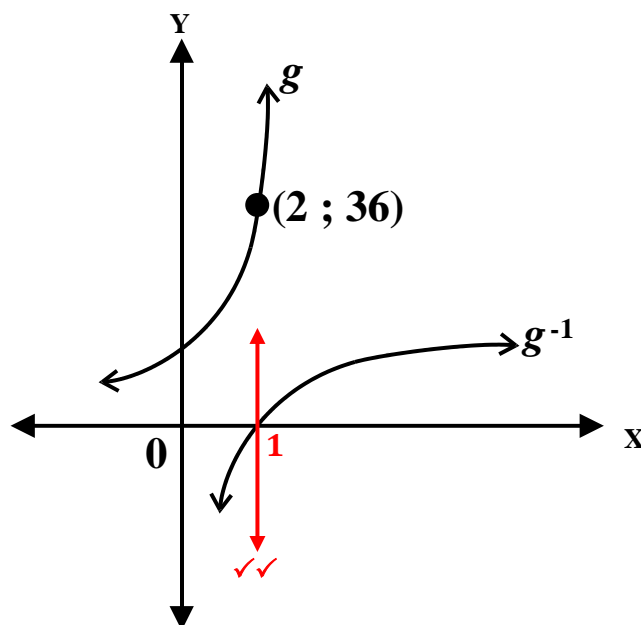
$$\therefore \text{RHS} = 3^{-2x} - 3^{2x}$$

$$\therefore \text{LHS} = \text{RHS}$$

$$\therefore [f(x)]^2 - [f(-x)]^2 = f(2x) - f(-2x)$$

Exercise F:

The graphs of $g(x) = k^x$, with $k > 0$ and $y = g^{-1}(x)$ is sketched below. The point $(2 ; 36)$ is a point on g .



- (1) Determine the value of k . (2)

$$g(x) = k^x \rightarrow y = k^x \rightarrow 36 = k^2 \quad \checkmark \quad \therefore k = 6 \quad \checkmark \quad (k > 0)$$

- (2) Write down the equation of g^{-1} in the form $y = \dots\dots$ (2)

$$g^{-1}: x = 6^y \quad \checkmark \quad \rightarrow \quad y = \log_6 x \quad \checkmark$$

- (3) For which value(s) of x will $g^{-1}(x) \leq 0$? (2)

$$0 < x \leq 1 \quad \text{or} \quad x \in (0 ; 1] \quad \checkmark$$

- (4) Write down the domain of h , for $h(x) = g^{-1}(x - 3)$. (1)

$$x > 3 \quad \text{or} \quad x \in (3 ; \infty) \quad \checkmark$$

- (5) Sketch the graph of the inverse of $y = 1$. See sketch above! (2)

$$\text{Inverse of } y = 1 \rightarrow x = 1$$

- (6) Is the inverse of $y = 1$ a function? Motivate your answer. (2)

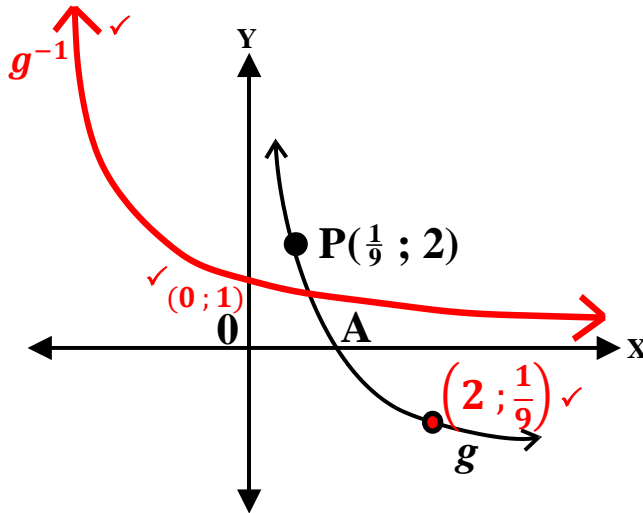
No, $x = 1$ is not a function, because the line is not a one-to-one function. $\checkmark\checkmark$

Exercise G:

(1) Given the graph of $g(x) = \log_{\frac{1}{3}} x$

A is the x -intercept of g .

$P(\frac{1}{9}; 2)$ is a point on g .



(1) Write down the coordinates of A. (1)

A(1; 0) ✓

(2) Sketch the graph of g^{-1} and indicate intercepts as well the coordinates of ONE other that will lie on the graph. **See graph above!** (3)

(3) Write down the domain of g^{-1} . (1)

\mathbb{R} or $x \in (-\infty; \infty)$ ✓