

# **Grade 12 – Textbook**

**(First edition – CAPS)**

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## Chapter A2

### Logarithms and function inverses

See grade 11 Functions and exponents for revision and background!

#### A2.1 Logarithms:

##### A2.1.1 Definition of a logarithm:

Logarithms are the inverses of exponents.

Ex. If  $2^5 = 32$  then  $\log_2 32 = 5$

$\therefore$  Per definition if  $y = \log_a x \Leftrightarrow x = a^y$  with  $a > 0 ; a \neq 1 \quad x > 0$

Remember: \*  $\log_a 1 = 0$  because  $a^0 = 1$

\* The natural logarithm is  $\log x \Leftrightarrow \log_{10} x$

\*  $\log_a a = 1$  because  $a^1 = a$

##### A2.1.2 Laws of logarithms:

For  $a > 0 ; a \neq 1 ; b > 0 ; b \neq 1 ; x > 0$  and  $y > 0$

- $\log_a x + \log_a y = \log_a xy$

- $\log_a x - \log_a y = \log_a \frac{x}{y}$

- $n \log_a x = \log_a x^n$

- $\log_a x = \frac{\log_b x}{\log_b a}$

**Ex. 1 Simplify: (Without using a calculator.)**

(a)  $\log_4 2 + \log_4 32$

$= \log_4(2 \times 32)$

$= \log_4(64)$

$= \log_4(4^3)$

$= 3\log_4(4)$

$= 3(1)$

$= 3$

(b)  $\log 200 - \log 2$

$= \log(200 \div 2)$

$= \log 100$

$= \log_{10} 10^2$

$= 2\log_{10} 10$

$= 2(1)$

$= 2$

(c)  $\log_3 36 \times \log_6 9$

$$\begin{aligned}
 &= \frac{\log 36}{\log 3} \times \frac{\log 9}{\log 6} \\
 &= \frac{\log 6^2}{\log 3} \times \frac{\log 3^2}{\log 6} \\
 &= \frac{2 \log 6}{\log 3} \times \frac{2 \log 3}{\log 6} \\
 &= \frac{2 \log 6}{\log 3} \times \frac{2 \log 3}{\log 6} \\
 &= 2 \times 2 \\
 &= 4
 \end{aligned}$$

(d)  $\log_4 16 + \log_3 \frac{1}{3} - \log_7 1$

$$\begin{aligned}
 &= \log_4 4^2 + \log_3 3^{-1} - 0 \\
 &= 2 \log_4 4 + (-1) \log_3 3 \\
 &= 2(1) - 1(1) \\
 &= 2 - 1 \\
 &= 1
 \end{aligned}$$

Ex. 2 If  $\log 3 = 0,477$  and  $\log 5 = 0,699$ , calculate:  
(Without using a calculator.)

(a)  $\log 45$

$$\begin{aligned}
 &= \log(9 \times 5) \\
 &= \log(3^2 \times 5) \\
 &= \log 3^2 + \log 5 \\
 &= 2 \log 3 + \log 5 \\
 &= 2 \times 0,477 + 0,699 \\
 &= 0,954 + 0,699 \\
 &= 1,653
 \end{aligned}$$

(b)  $\log 30$

$$\begin{aligned}
 &= \log(3 \times 10) \\
 &= \log 3 + \log 10 \\
 &= \log 3 + \log 10 \\
 &= 0,477 + 1 \\
 &= 1,477
 \end{aligned}$$

Ex. 3 Solve for  $x$ : (Without using a calculator.)

(a)  $\log x + \log(x + 3) = 1$

$$\begin{aligned}
 \therefore \log_{10} x(x + 3) &= 1 \\
 \therefore 10^1 &= x^2 + 3x \\
 \therefore 0 &= x^2 + 3x - 10 \\
 \therefore 0 &= (x + 5)(x - 2) \\
 \therefore x &= -5 \text{ or } x = 2 \\
 \text{but } x &\neq -5, \text{ because } x > 0
 \end{aligned}$$

(b)  $\log_3(x + 4) - \log_3 x = \log_3 5$

$$\begin{aligned}
 \therefore \log_3 \frac{(x+4)}{x} &= \log_3 5 \\
 \therefore \log_3 \frac{(x+4)}{x} &= \log_3 5 \\
 \therefore \frac{(x+4)}{x} &= 5 \quad [\text{Per definition}] \\
 \therefore x + 4 &= 5x \\
 \therefore x - 5x &= -4 \\
 \therefore -4x &= -4 \\
 \therefore x &= 1
 \end{aligned}$$

Ex. 4 Solve for  $x$ : (Use a calculator and give your answer correct to 2 decimals.)

(a)  $3^x = 7$

$$\begin{aligned}
 \therefore \log_3 7 &= x \\
 \therefore x &= \frac{\log 7}{\log 3} \\
 \therefore x &\approx 1,77
 \end{aligned}$$

(b)  $1,3 = 2^{x-3}$

$$\begin{aligned}
 \therefore \log_2 1,3 &= x - 3 \\
 \therefore x - 3 &= \frac{\log 1,3}{\log 2} \\
 \therefore x - 3 &= 0,3785 \dots \\
 \therefore x &\approx 3,38
 \end{aligned}$$

Exercise 1:

(1) Write the following in logarithmic form:

(a)  $7^3 = 343$

(b)  $x = \left(\frac{1}{2}\right)^2$

(c)  $y = 2^{x+1}$

(d)  $2^{\log x} = 5$

(2) Write the following in exponential form:

(a)  $\log_2 32 = 5$

(b)  $\log y = k$

(c)  $m = \log_3 k$

(d)  $\log_3 \frac{1}{27} = -3$

(3) Write the following as separate logarithms with base 10 if  $\{x; y; t; p\} > 0$ :

(a)  $\log \frac{xy}{p}$

(b)  $\log_t p^2 t$

(4) Write the following as a single logarithm if  $\{x; y; t; p\} > 0$ :

(a)  $\log t - \log y + 2 \log p$

(b)  $\log_2(x-2) - \log_2(x+1) - \log_2 x$

(5) Simplify without using a calculator:

(a)  $\log 25 + \log 8 - \log 2$

(b)  $\log_2 16 + 3 \log_3 \left(\frac{1}{9}\right) - \log_{15} 1$

(c)  $\frac{\log 32 - \log 243}{\log 3 - \log 2}$

(d)  $\frac{\log_5 27 + \log_5 9}{\log_5 \sqrt{3}}$

(e)  $\log 8000 - \log 8$

(f)  $\frac{1}{2} \log_4 16 + \log_{0,2} 0,04 - \log_3 \sqrt{27} - \log 25 \times \log_5 1$

(6) Solve for  $x$ : [Where necessary, round off correct to 2 decimals.]

(a)  $\log_4 2x = 3$

(b)  $\log_3(x+2) + \log_3 x = 1$

(c)  $\log_2(2x+12) - 2 \log_2 x = 1$

(d)  $7^{3x} = 14$

(7) Write the following in terms of  $m$  and/or  $n$  if  $\log 6 = m$  and  $\log 3 = n$ :

(a)  $\log 18$

(b)  $\log_{27} 36$

(c)  $\log 300$

(d)  $\log 20$

## A2.2 Inverses:

The rule for the reflection in the line  $x = y$  is:  $(x ; y) \Leftrightarrow (y ; x)$

This reflection in the line  $y = x$  is referred to as the inverse  $\Leftrightarrow$  it means that the  $x$  and  $y$  swap places!

The inverse of  $f(x)$  is written as  $f^{-1}(x)$ .

**Ex. 5 Determine  $f^{-1}(x)$  in each of the following in the form  $f^{-1}(x) = \dots$  :**

(a)  $f(x) = 5x^2$

$\therefore$  For  $f$ :  $y = 5x^2$

$\therefore$  For  $f^{-1}$ :  $x = 5y^2$

$\therefore \frac{x}{5} = y^2$

$\therefore y = \pm \sqrt{\frac{x}{5}}$

$\therefore f^{-1}(x) = \pm \sqrt{\frac{x}{5}}$

(b)  $f: x \rightarrow \frac{3}{x+2}$

$\therefore$  For  $f$ :  $y = \frac{3}{x+2}$

$\therefore$  For  $f^{-1}$ :  $x = \frac{3}{y+2}$

$\therefore y + 2 = \frac{3}{x}$

$\therefore y = \frac{3}{x} - 2$

$\therefore f^{-1}(x) = \frac{3}{x} - 2$

### Exercise 2:

(1) Determine  $f^{-1}(x)$  in each of the following and write it in the form  $f^{-1}(x) = \dots\dots$

(a)  $f(x) = 3x - 4$

(b)  $f(x) = 5^x$

(c)  $f(x) = -2x^2$

(d)  $f(x) = \log_{0,5} x$

(2) Determine  $g^{-1}(x)$  in each of the following and write it in the form  $g^{-1}: x \rightarrow \dots\dots$

(a)  $g: x \rightarrow \frac{x}{4}$

(b)  $g: x \rightarrow \log_3 x$

(c)  $g: x \rightarrow 3^{x+1}$

(d)  $g: x \rightarrow -0,5x$

(3) Determine  $h$  in each of the following and write it in the form  $h(x) = \dots\dots$

(a)  $h^{-1}(x) = \log_7 x$

(b)  $h^{-1}(x) = \frac{x-2}{3}$

(c)  $h^{-1}(x) = \frac{1}{4}x^2$

(d)  $h^{-1}(x) = \log x$

(4) Consider the following:  $p(x) = \{(1; 7); (2; 8); (3; 9); (4; 10)\}$

(a) Is  $p$  a function? Motivate your answer.

(b) Write down the range of  $p^{-1}(x)$ .

(5) Explain the difference between  $f^{-1}(x)$  and  $(f(x))^{-1}$ .

## A2.3 Graphs of inverses:

### A2.3.1 Graphs of inverses of the straight line:

See grade 11 Linear Functions for revision and background!

If the function  $f(x) = mx + c$  is given, the inverse will be obtained as follows:

$$f(x) = mx + c \Leftrightarrow y = mx + c$$

$\therefore$  For the inverse the  $x$  and  $y$  swap places:  $x = my + c$

$$\Rightarrow y = \frac{x - c}{m} \quad [\text{Make } y \text{ the subject!}]$$

$$\therefore f^{-1}(x) = \frac{x - c}{m} \Rightarrow \text{Inverse function}$$

**Ex. 6 Given:**  $g(x) = 2x - 4$

(a) Determine  $g^{-1}(x) = \dots$

(b) Sketch  $g(x)$  and  $g^{-1}(x)$  on the same system of axes.

(a)  $g(x): y = 2x - 4 \Leftrightarrow \therefore g^{-1}(x): x = 2y - 4$

$$\therefore 2y = x + 4$$

$$\therefore y = \frac{1}{2}x + 2 \quad \therefore g^{-1}(x) = \frac{1}{2}x + 2$$

(b) For  $g(x)$ :  $x$ -intercept ( $y = 0$ )  $y$ -intercept ( $x = 0$ )

$$2x - 4 = 0$$

$$y = 2(0) - 4$$

$$\therefore x = 2$$

$$\therefore y = -4$$

$$\therefore (2; 0) \quad \text{and} \quad (0; -4)$$

For  $g^{-1}(x)$ :  $x$  and  $y$  of  $g(x)$  swap places:

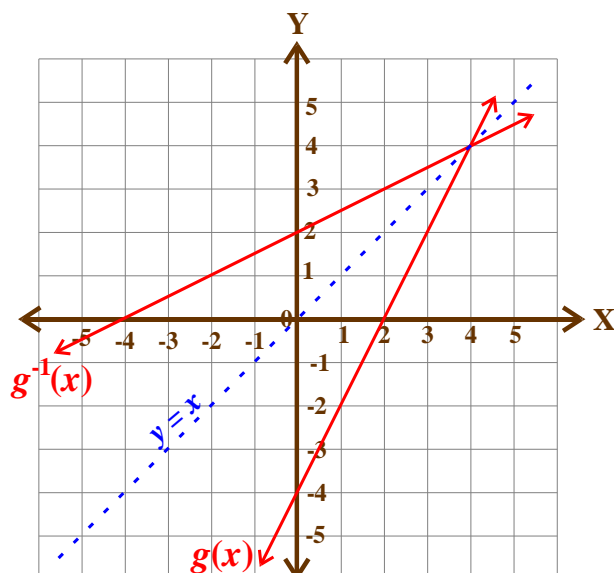
$y$ -intercept ( $x = 0$ )

$x$ -intercept ( $y = 0$ )

$$\therefore (0; 2)$$

and

$$(-4; 0)$$



## A2.3.2 Graphs of inverses of the parabola:

See grade 11 Quadratic Functions for revision and background!

Ex. 7 Given:  $g(x) = 2x^2$  with  $x \geq 0$

(a) Determine  $g^{-1}(x) = \dots$

(b) Sketch  $g(x)$  and  $g^{-1}(x)$  on the same system of axes.

(c) Write down the domain of  $g^{-1}(x)$ .

$$(a) \quad g(x): y = 2x^2 \text{ with } x \geq 0 \quad \Leftrightarrow \quad \therefore \quad g^{-1}(x): x = 2y^2 \text{ with } y \geq 0$$

$$\therefore \quad y^2 = \frac{x}{2}$$

$$\therefore \quad y = \pm \sqrt{\frac{x}{2}}$$

$$\text{but } y \geq 0$$

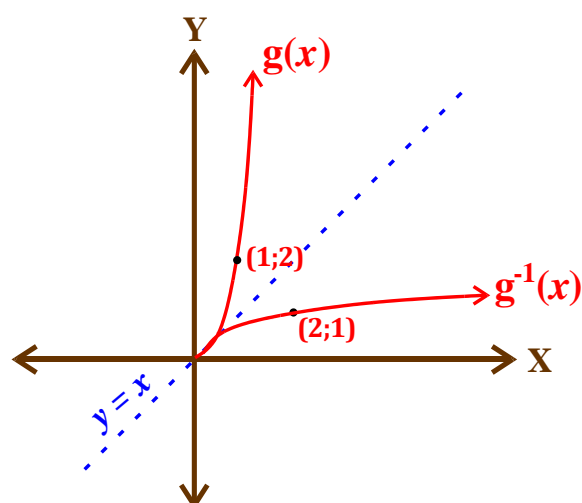
$$\therefore \quad g^{-1}(x) = +\sqrt{\frac{x}{2}}$$

(b) For  $g(x)$ : Use a table, because the  $x$ -and- $y$  intercepts and the turning point is  $(0; 0)$ .

$x$	0	1	2	$x \geq 0$
$y$	0	2	8	

For  $g^{-1}(x)$ :  $x$  and  $y$  of  $g(x)$  swap place:

$x$	0	2	8
$y$	0	1	2



(c)  $D_{g^{-1}}: x \geq 0$

### A2.3.3 Graphs of inverses of the exponential function:

See grade 11 Exponential Functions for revision and background!

If the function  $f(x) = a^x$  is given, the inverse will be obtained as follows:

$$f(x) = a^x \Leftrightarrow y = a^x$$

$\therefore$  For the inverse, the  $x$  and  $y$  swap places:  $x = a^y$

$$\Rightarrow y = \log_a x \quad \text{[Make } y \text{ the subject!]}$$

$\therefore$  The inverse of an exponential function is a logarithmic function.

**Ex. 8 Given:  $g(x) = 2^x$**

(a) Determine  $g^{-1}(x) = \dots$

(b) Sketch  $g(x)$  and  $g^{-1}(x)$  on the same system of axes.

(c) Write down the equation of the asymptote of  $g^{-1}(x)$ .

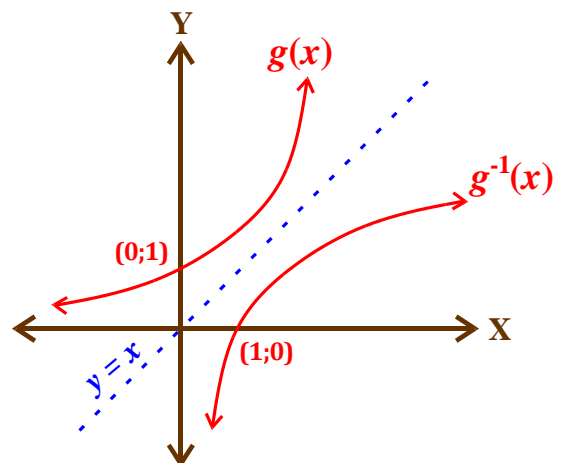
$$\begin{aligned} \text{(a) } g(x): y = 2^x & \Leftrightarrow \therefore g^{-1}(x): x = 2^y \\ & \therefore y = \log_2 x \\ & \therefore g^{-1}(x) = \log_2 x \end{aligned}$$

(b) For  $g(x)$ :

$x$	-1	0	1
$y$	$\frac{1}{2}$	1	2

For  $g^{-1}(x)$ :  $x$  and  $y$  of  $g(x)$  swap places:

$x$	$\frac{1}{2}$	1	2
$y$	-1	0	1



(c) Asymptote of  $g^{-1}(x)$ :

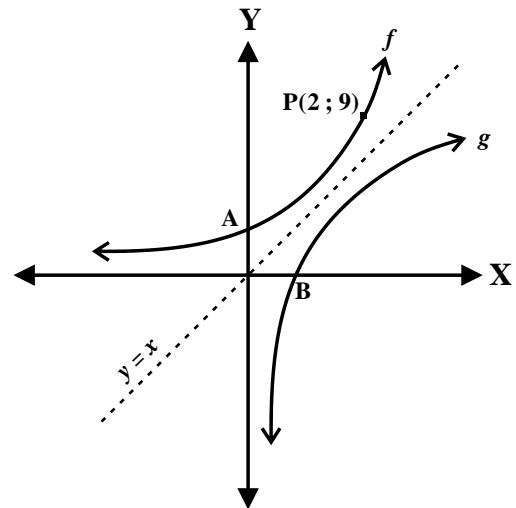
$$x = 0$$



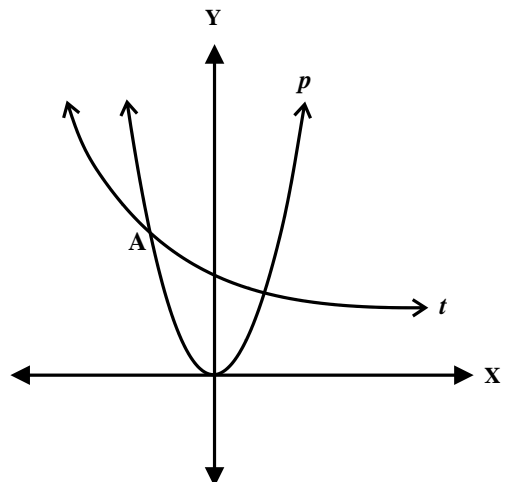
Exercise 3:

- (1) (a) Sketch:  $g(x) = x^2 + 1$  for  $x \leq 0$ .  
 (b) Determine  $g^{-1}$  and write it in the form  $g^{-1}(x) = \dots\dots$   
 (c) Sketch  $g^{-1}(x)$  on the same system of axes as  $g(x)$ .  
 (d) Write down the range of  $g^{-1}(x)$ .
- (2) Given:  $h(x) = 2^{-x}$   
 (a) Determine  $h^{-1}$  and write it in the form  $h^{-1}(x) = \dots\dots$   
 (b) Sketch  $h$  and  $h^{-1}$  on the same system of axes.  
 (c) Write down the domain of  $h^{-1}(x)$ .  
 (d) If  $p$  is the reflection of  $h$  in the  $y$ -axis, determine the equation of  $p$  and write it in the form  $p(x) = \dots\dots$   
 (e) Determine  $p^{-1}$  and write it in the form  $p^{-1}(x) = \dots\dots$

- (3) Given:  $f(x) = a^x$  and  $g(x)$   
 with  $P(2; 9)$ .  
 (a) Determine the value of  $a$ .  
 (b) Give the coordinates of A.  
 (c) Determine the equation of  $g(x)$ ,  
 if  $g(x)$  is the mirror image of  $f(x)$   
 in the line  $y = x$ .  
 (d) Give the coordinates of B.  
 (e) For which values of  $x$  will  $g(x)$  be defined?  
 (f) Write down the equation of the asymptote of  $g(x)$ .



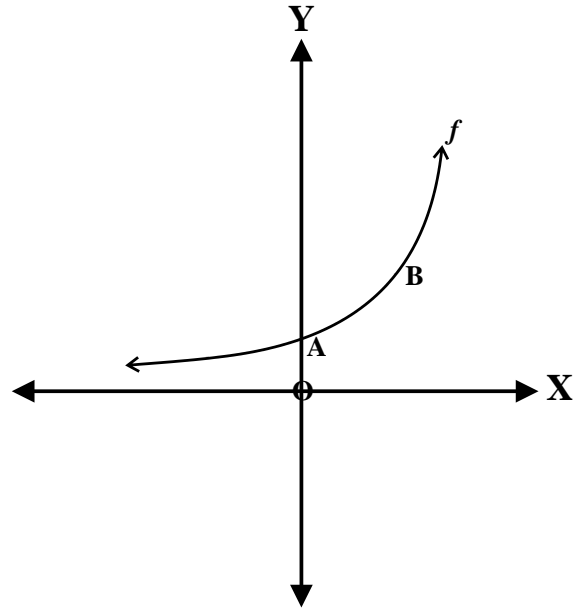
- (4) Given:  $t(x) = a^x$  and  $p(x) = bx^2$   
 met  $A(-2; 4)$ .  
 (a) Determine the values of  $a$  and  $b$ .  
 (b) Write down the following:  $t^{-1}(x) = \dots\dots$   
 (c) Write down the following:  $p^{-1}(x) = \dots\dots$   
 (d) Explain why  $p^{-1}(x)$  is not a function.  
 (e) Determine  $x$  for which  $t^{-1}(x) \geq 0$ .  
 (f) Calculate:  $t^{-1}(0,25) + p(3)$



- (5) The graph of  $f(x) = a^x$  is sketched alongside.

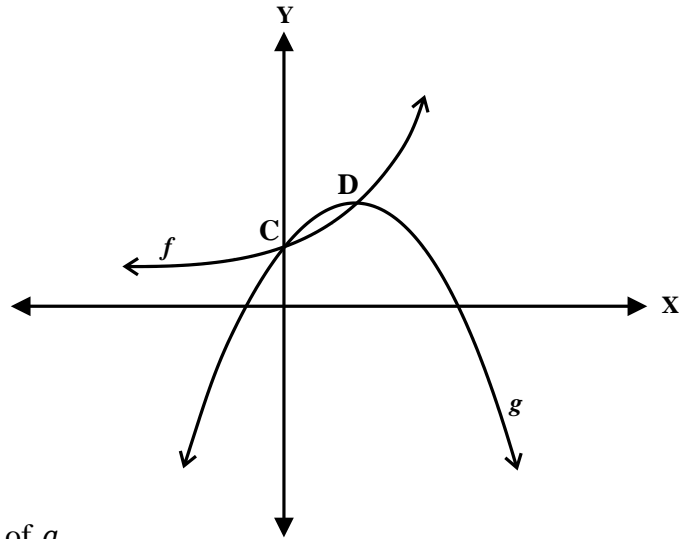
The point  $B(3; 8)$  lies on the graph of  $f$ .

- Show that  $a = 2$ .
- Write down the coordinates of A.
- Write down the equation of  $f^{-1}(x)$  in the form  $f^{-1}(x) = \dots$
- Sketch the graph of  $f^{-1}$ .  
Show the  $x$ -intercept and ONE other point.
- For which values of  $x$  will  $f^{-1}(x) = f(x)$ ?
- Write down the equation of  $g$  if  $g$  is the reflection of  $f$  in the  $y$ -axis.
- Write down the equation of  $h$  if  $h$  is the reflection of  $f^{-1}$  in the  $x$ -axis.
- Are  $g$  and  $h$  one another's inverse? Motivate your answer.
- For which values of  $x$  will  $f^{-1}(x) \geq 0$ ?
- Calculate:  $f^{-1}(2) + f(-2)$



- (6) On the right is the graphs of  $f(x) = 2^x$  and  $g(x) = -(x - 1)^2 + b$ , with  $b$  as a constant value. The graphs of  $f$  and  $g$  intersect on the  $y$ -axis at C. D is the turning point of  $g$ .

- Show that  $b = 2$ .
- Write down the coordinates of the turning point of  $g$ .
- Write down the equation of  $f^{-1}(x)$  in the form  $y = \dots\dots$
- Sketch the graph of  $f^{-1}$  on the same graph as given above.  
Show on your graph the  $x$ -intercept and the coordinates of one other point.
- Write down the equation of  $h$  if  $h(x) = g(x + 1) - 2$ .
- How can the domain of  $h$  be restricted so that  $h^{-1}$  will be a function?
- Determine the maximum value of  $2^2 - (x - 1)^2$ .



## REVISION FROM PAST PAPERS:

### Exercise A:

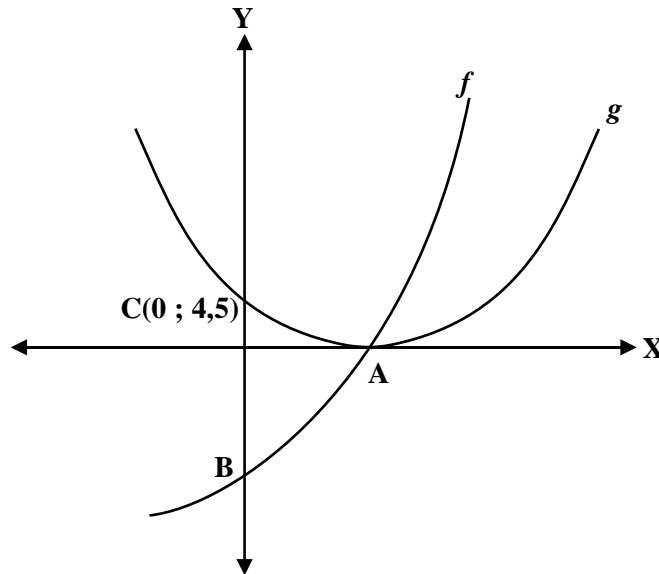
Consider the function  $f(x) = \left(\frac{1}{3}\right)^x$

- (1) Is  $f$  an increasing or decreasing function? Give a reason for your answer. (2)
- (2) Calculate  $f^{-1}(x)$  in the form  $y = \dots\dots\dots$  (2)
- (3) Write down the equation of the asymptote of  $f(x) - 5$ . (1)
- (4) Describe the transformation of  $f$  to  $g$  if  $g(x) = \log_3 x$ . (2)

### Exercise B:

The graphs of  $f(x) = 2^x - 8$  and  $g(x) = ax^2 + bx + c$

sketched below. B and C(0 ; 4,5) are the  $y$ -intercepts of the graphs of  $f$  and  $g$  respectively. The two graphs intersect at A, which is the turning point of the graph of  $g$  and the  $x$ -intercept of the graphs of  $f$  and  $g$ .



- (1) Determine the coordinates of A and B. (4)
- (2) Write down the equation of the asymptote of graph  $f$ . (1)
- (3) Determine the equation of  $h$  if  $h(x) = f(2x) + 8$ . (2)
- (4) Determine the equation of  $h^{-1}$  in the form  $y = \dots\dots\dots$  (2)
- (5) Write down the equation of  $p$ , if  $p$  is the reflection of  $h^{-1}$  in the  $x$ -axis. (1)
- (6) Calculate  $\sum_{k=0}^3 g(k) - \sum_{k=4}^5 g(k)$ . Show ALL calculations. (4)

Exercise C:

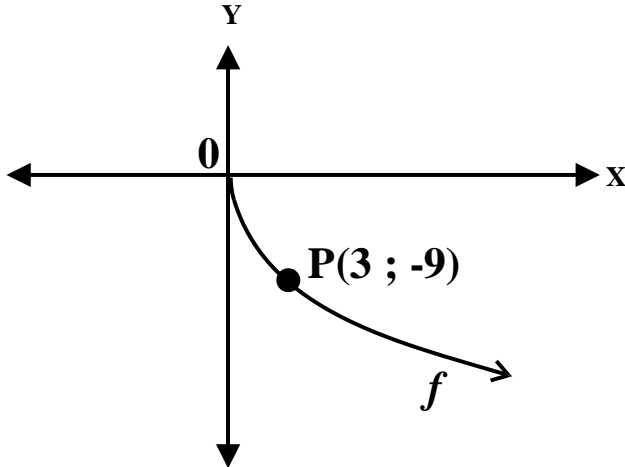
Given:  $f(x) = 3^x$

- (1) Determine an equation for  $f^{-1}$  in the form  $f^{-1}(x) = \dots$  (1)
- (2) Sketch the graphs of  $f$  and  $f^{-1}$ , clearly showing ALL intercepts with the axes. (4)
- (3) Write down the domain of  $f^{-1}$ . (2)
- (4) For which values of  $x$  will  $f(x) \cdot f^{-1}(x) \leq 0$ ? (2)
- (5) Write down the range of  $h(x) = 3^{-x} - 4$ . (2)
- (6) Write down an equation for  $g$  if the graph of  $g$  is the image of the graph of  $f$  after  $f$  has been translated two units to the right and reflected about the  $x$ -axis. (2)

Exercise D:

The graph of  $f(x) = -\sqrt{27x}$  for  $x \geq 0$  is sketched below.

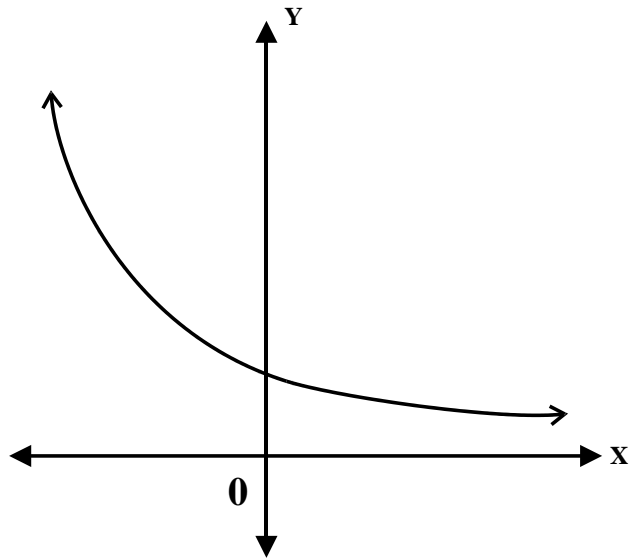
The point  $P(3; -9)$  lies on the graph of  $f$ .



- (1) Use the graph to determine the values of  $x$  for which  $f(x) \geq -9$ . (2)
- (2) Write down the equation of  $f^{-1}$  in the form  $y = \dots$ . (3)  
Include ALL restrictions.
- (3) Sketch  $f^{-1}$ , the inverse of  $f$ . Indicate the intercept(s) with the axes and the coordinates of ONE other point. (3)
- (4) Describe the transformation of  $f$  to  $g$  if  $g(x) = \sqrt{27x}$  for  $x \geq 0$ . (1)

Exercise E:

The graph of  $f(x) = \left(\frac{1}{3}\right)^x$  is sketched on the right.

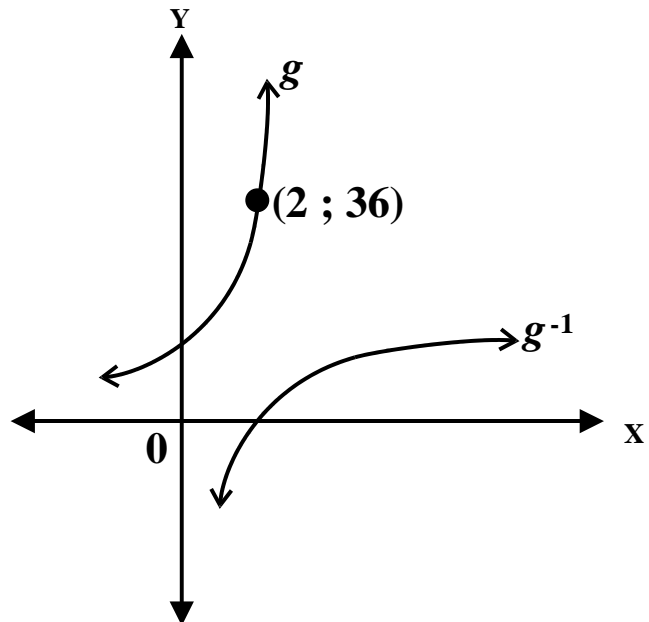


- (1) Write down the domain of  $f$ . (1)
- (2) Write down the equation of the asymptote of  $f$ . (1)
- (3) Write down the equation of  $f^{-1}$  in the form  $y = \dots$ . (2)
- (4) Sketch the graph of  $f^{-1}$ . Indicate the  $x$ -intercept and the coordinates of ONE other point. (3)
- (5) Write down the equation of the asymptote of  $f^{-1}(x + 2)$ . (2)
- (6) Prove that:  $[f(x)]^2 - [f(-x)]^2 = f(2x) - f(-2x)$  for all values of  $x$ . (3)

Exercise F:

The graphs of  $g(x) = k^x$ , with  $k > 0$  and  $y = g^{-1}(x)$  is sketched on the right.

The point  $(2 ; 36)$  is a point on  $g$ .



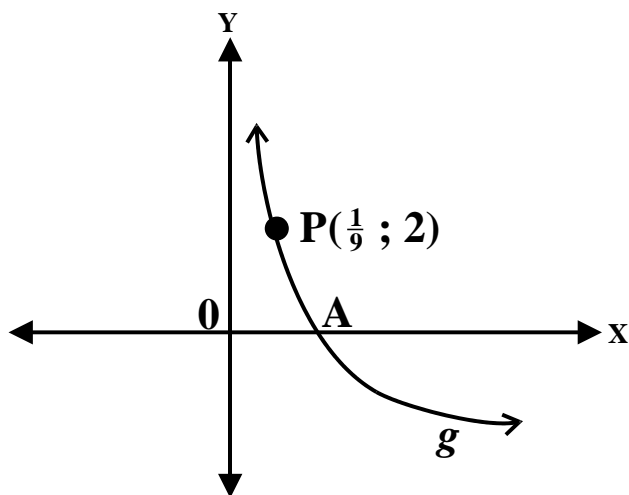
- (1) Determine the value of  $k$ . (2)
- (2) Write down the equation of  $g^{-1}$  in the form  $y = \dots$ . (2)
- (3) For which value(s) of  $x$  will  $g^{-1}(x) \leq 0$ ? (2)
- (4) Write down the domain of  $h$ , for  $h(x) = g^{-1}(x - 3)$ . (1)
- (5) Sketch the graph of the inverse of  $y = 1$ . (2)
- (6) Is the inverse of  $y = 1$  a function? Motivate your answer. (2)

Exercise G:

Given the graph of  $g(x) = \log_{\frac{1}{3}} x$

A is the  $x$ -intercept of  $g$ .

$P\left(\frac{1}{9}; 2\right)$  is a point on  $g$ .



- (1) Write down the coordinates of A. (1)
- (2) Sketch the graph of  $g^{-1}$  and indicate intercepts as well the coordinates of ONE other that will lie on the graph. (3)
- (3) Write down the domain of  $g^{-1}$ . (1)