Grade 12 – Textbook
(First edition – CAPS)

CONTENTS:

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Chapter A2
Logarithms and function inverses

See grade 11 Functions and exponents for revision and background!

A2.1 Logarithms:
A2.1.1 Definition of a logarithm:

Logarithms are the inverses of exponents.

Ex. If $2^5 = 32$ then $\log_2 32 = 5$

$\therefore$ Per definition if $y = \log_a x \iff x = a^y$ with $a > 0\ ;\ a \neq 1 \quad x > 0$

Remember: * $\log_a 1 = 0$ because $a^0 = 1$

* The natural logarithm is $\log x \iff \log_{10} x$

* $\log_a a = 1$ because $a^1 = a$

A2.1.2 Laws of logarithms:

For $a > 0\ ;\ a \neq 1\ ;\ b > 0\ ;\ b \neq 1\ ;\ x > 0$ and $y > 0$

- $\log_a x + \log_a y = \log_a xy$
- $\log_a x - \log_a y = \log_a \frac{x}{y}$
- $n \log_a x = \log_a x^n$
- $\log_a x = \frac{\log_b x}{\log_b a}$

Ex. 1 Simplify: (Without using a calculator.)

(a) $\log_4 2 + \log_4 32$

$= \log_4(2 \times 32)$

$= \log_4(64)$

$= \log_4(4^3)$

$= 3\log_4(4)$

$= 3(1)$

$= 3$

(b) $\log 200 - \log 2$

$= \log(200 \div 2)$

$= \log 100$

$= \log_{10} 10^2$

$= 2\log_{10} 10$

$= 2(1)$

$= 2$

(c) $\log_3 36 \times \log_6 9$

$= \frac{\log 36}{\log 3} \times \frac{\log 9}{\log 6}$

$= \frac{\log 6^2}{\log 3} \times \frac{\log 3^2}{\log 6}$

$= \frac{2\log 6}{\log 3} \times \frac{2\ \log 3}{\log 6}$

$= \frac{2\cancel{\log 6}}{\cancel{\log 3}} \times \frac{2\ \cancel{\log 3}}{\cancel{\log 6}}$

$= 2 \times 2$

$= 4$

(d) $\log_4 16 + \log_3 \frac{1}{3} - \log_7 1$

$= \log_4 4^2 + \log_3 3^{-1} - 0$

$= 2\log_4 4 + (-1)\log_3 3$

$= 2(1) - 1(1)$

$= 2 - 1$

$= 1$

Ex. 2 If $\log 3 = 0,477$ and $\log 5 = 0,699$, calculate:
(Without using a calculator.)

(a) $\log 45$

$= \log(9 \times 5)$

$= \log(3^2 \times 5)$

$= \log 3^2 + \log 5$

$= 2\ \log 3 + \log 5$

$= 2 \times 0,477 + 0,699$

$= 0,954 + 0,699$

$= 1,653$

(b) $\log 30$

$= \log(3 \times 10)$

$= \log 3 + \log 10$

$= \log 3 + \log 10$

$= 0,477 + 1$

$= 1,477$

Ex. 3 Solve for x : (Without using a calculator.)

(a) $\log x + \log(x + 3) = 1$

$\therefore \log_{10} x(x + 3) = 1$

$\therefore 10^1 = x^2 + 3x$

$\therefore 0 = x^2 + 3x - 10$

$\therefore 0 = (x + 5)(x - 2)$

$\therefore x = -5$ or $x = 2$

but $x \neq -5$, because $x > 0$

(b) $\log_3(x + 4) - \log_3 x = \log_3 5$

$\therefore \log_3 \frac{(x+4)}{x} = \log_3 5$

$\therefore \log_3 \frac{(x+4)}{x} = \log_3 5$

$\therefore \frac{(x+4)}{x} = 5$ [Per definition]

$\therefore x + 4 = 5x$

$\therefore x - 5x = -4$

$\therefore -4x = -4$

$\therefore x = 1$

Ex. 4 Solve for x : (Use a calculator and give your answer correct to 2 decimals.)

(a) $3^x = 7$

$\therefore \log_3 7 = x$

$\therefore x = \frac{\log 7}{\log 3}$

$\therefore x \approx 1,77$

(b) $1,3 = 2^{x-3}$

$\therefore \log_2 1,3 = x - 3$

$\therefore x - 3 = \frac{\log 1,3}{\log 2}$

$\therefore x - 3 = 0,3785 \ldots$

$\therefore x \approx 3,38$

Exercise 1:

(1) Write the following in logarithmic form:

(a) $7^3 = 343$ (b) $x = \left(\frac{1}{2}\right)^2$

(c) $y = 2^{x+1}$ (d) $2^{\log x} = 5$

(2) Write the following in exponential form:

(a) $\log_2 32 = 5$ (b) $\log y = k$

(c) $m = \log_3 k$ (d) $\log_3 \frac{1}{27} = -3$

(3) Write the following as separate logarithms with base 10 if $\{x\,;y\,;t\,;p\} > 0$:

(a) $\log \dfrac{xy}{p}$ (b) $\log_t p^2\, t$

(4) Write the following as a single logarithm if $\{x\,;y\,;t\,;p\} > 0$:

(a) $\log t - \log y + 2\log p$ (b) $\log_2(x-2) - \log_2(x+1) - \log_2 x$

(5) Simplify without using a calculator:

(a) $\log 25 + \log 8 - \log 2$

(b) $\log_2 16 + 3\log_3\left(\frac{1}{9}\right) - \log_{15} 1$

(c) $\dfrac{\log 32 - \log 243}{\log 3 - \log 2}$

(d) $\dfrac{\log_5 27 + \log_5 9}{\log_5 \sqrt{3}}$

(e) $\log 8\,000 - \log 8$

(f) $\frac{1}{2}\log_4 16 + \log_{0,2} 0{,}04 - \log_3 \sqrt{27} - \log 25 \times \log_5 1$

(6) Solve for x: [Where necessary, round off correct to 2 decimals.]

(a) $\log_4 2x = 3$

(b) $\log_3(x + 2) + \log_3 x = 1$

(c) $\log_2(2x + 12) - 2\log_2 x = 1$

(d) $7^{3x} = 14$

(7) Write the following in terms of m and/or n if $\log 6 = m$ and $\log 3 = n$:

(a) $\log 18$ (b) $\log_{27} 36$

(c) $\log 300$ (d) $\log 20$

A2.2 Inverses:

The rule for the reflection in the line $x = y$ is: $(x\,;\,y) \iff (y\,;\,x)$

This reflection in the line $y = x$ is referred to as the inverse $\iff$ it means that the x and y swap places!

The inverse of $f(x)$ is written as $f^{-1}(x)$.

Ex. 5 Determine $f^{-1}(x)$ in each of the following in the form $f^{-1}(x) = \ldots$:

(a) $f(x) = 5x^2$

$\therefore$ **For f:** $y = 5x^2$

$\therefore$ **For f^{-1}:** $x = 5y^2$

$\therefore$ $\frac{x}{5} = y^2$

$\therefore$ $y = \pm\sqrt{\frac{x}{5}}$

$\therefore$ $f^{-1}(x) = \pm\sqrt{\frac{x}{5}}$

(b) $f: x \to \frac{3}{x+2}$

$\therefore$ **For f:** $y = \frac{3}{x+2}$

$\therefore$ **For f^{-1}:** $x = \frac{3}{y+2}$

$\therefore$ $y + 2 = \frac{3}{x}$

$\therefore$ $y = \frac{3}{x} - 2$

$\therefore$ $f^{-1}(x) = \frac{3}{x} - 2$

Exercise 2:

(1) Determine $f^{-1}(x)$ in each of the following and write it in the form $f^{-1}(x) = \ldots\ldots.$

(a) $f(x) = 3x - 4$ (b) $f(x) = 5^x$

(c) $f(x) = -2x^2$ (d) $f(x) = \log_{0,5} x$

(2) Determine $g^{-1}(x)$ in each of the following and write it in the form $g^{-1}: x \to \ldots\ldots.$

(a) $g : x \to \frac{x}{4}$ (b) $g : x \to \log_3 x$

(c) $g : x \to 3^{x+1}$ (d) $g : x \to -0{,}5x$

(3) Determine h in each of the following and write it in the form $h(x) = \ldots\ldots.$

(a) $h^{-1}(x) = \log_7 x$ (b) $h^{-1}(x) = \frac{x-2}{3}$

(c) $h^{-1}(x) = \frac{1}{4}x^2$ (d) $h^{-1}(x) = \log x$

(4) Consider the following: $p(x) = \{(1;7)\,;(2;8)\,;(3:9)\,;(4;10)\}$

(a) Is p a function? Motivate your answer.

(b) Write down the range of $p^{-1}(x)$.

(5) Explain the difference between $f^{-1}(x)$ and $(f(x))^{-1}$.

A2.3 Graphs of inverses:
A2.3.1 Graphs of inverses of the straight line:

See grade 11 Linear Functions for revision and background!

If the function $f(x) = mx + c$ is given, the inverse will obtained as follow:

$f(x) = mx + c \iff y = mx + c$

$\therefore$ For the inverse the x and y swap place: $x = my + c$

$\implies y = \frac{x - c}{m}$ [Make y the subject!]

$\therefore f^{-1}(x) = \frac{x - c}{m} \implies$ Inverse function

Ex. 6 Given: $g(x) = 2x - 4$

(a) Determine $g^{-1}(x) = \ldots..$

(b) Sketch $g(x)$ and $g^{-1}(x)$ on the same system of axes.

(a) $g(x)$: $y = 2x - 4 \iff \therefore g^{-1}(x)$: $x = 2y - 4$

$\therefore 2y = x + 4$

$\therefore y = \frac{1}{2}x + 2 \qquad \therefore g^{-1}(x) = \frac{1}{2}x + 2$

(b) For $g(x)$: x-intercept $(y = 0)$ $\qquad$ y-intercept $(x = 0)$

$2x - 4 = 0 \qquad y = 2(0) - 4$

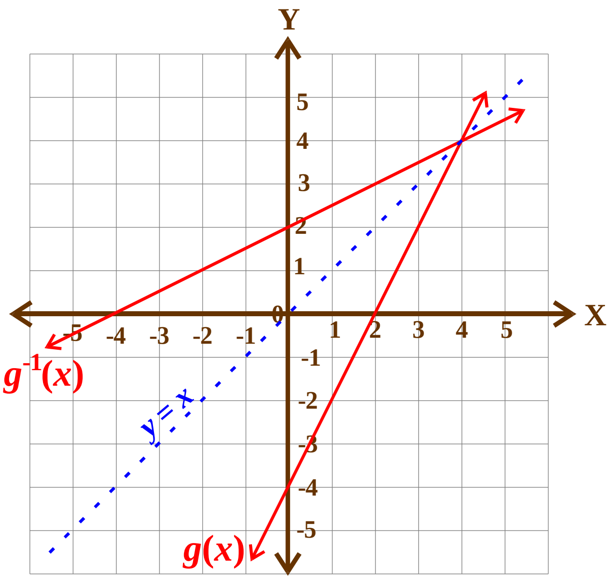
$\therefore x = 2 \qquad \therefore y = -4$

$\therefore (2\,;\,0)$ and $(0\,;\,-4)$

For $g^{-1}(x)$: x and y of $g(x)$ swap place:

y-intercept $(x = 0)$ $\qquad$ x-intercept $(y = 0)$

$\therefore (0\,;\,2)$ and $(-4\,;\,0)$



A2.3.2 Graphs of inverses of the parabola:

See grade 11 Quadratic Functions for revision and background!

Ex. 7 Given: $g(x) = 2x^2$ with $x \geq 0$

(a) Determine $g^{-1}(x) = \ldots..$

(b) Sketch $g(x)$ and $g^{-1}(x)$ on the same system of axes.

(c) Write down the domain of $g^{-1}(x)$.

(a) $g(x)$: $y = 2x^2$ with $x \geq 0$ $\Leftrightarrow$ $\therefore$ $g^{-1}(x)$: $x = 2y^2$ with $y \geq 0$

$\therefore \quad y^2 = \frac{x}{2}$

$\therefore \quad y = \pm\sqrt{\frac{x}{2}}$

but $y \geq 0$

$\therefore \quad g^{-1}(x) = +\sqrt{\frac{x}{2}}$

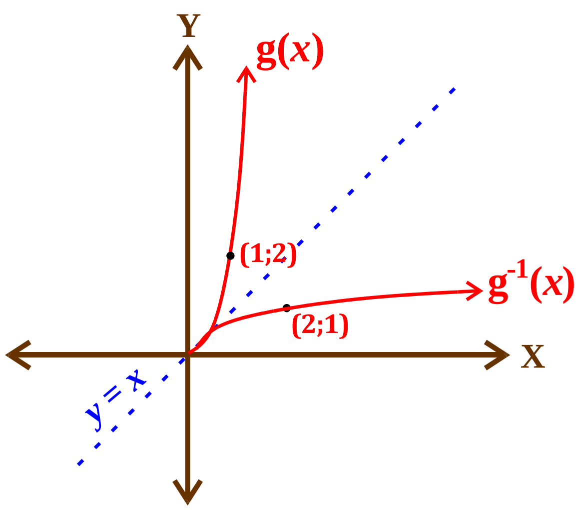
(b) For $g(x)$: Use a table, because the x-and-y intercepts and the turning point is $(0\,;\,0)$.

x	0	1	2
y	0	2	8

$x \geq 0$

For $g^{-1}(x)$: x and y of $g(x)$ swap place:

x	0	2	8
y	0	1	2



(c) $D_{g^{-1}}$: $x \geq 0$

A2.3.3 <u>Graphs of inverses of the exponential function</u>:

See grade 11 Exponential Functions for revision and background!

If the function $f(x) = a^x$ is given, the inverse will obtained as follow:

$f(x) = a^x \iff y = a^x$

$\therefore$ For the inverse, the x and y swap places: $x = a^y$

$\implies y = \log_a x$ [Make y the subject!]

$\therefore$ The inverse of an exponential function is a logarithmic function.

Ex. 8 Given: $g(x) = 2^x$

(a) Determine $g^{-1}(x) = \ldots\ldots$

(b) Sketch $g(x)$ and $g^{-1}(x)$ on the same system of axes.

(c) Write down the equation of the asymptote of $g^{-1}(x)$.

(a) $g(x)$: $y = 2^x \iff \therefore g^{-1}(x)$: $x = 2^y$

$\therefore y = \log_2 x$

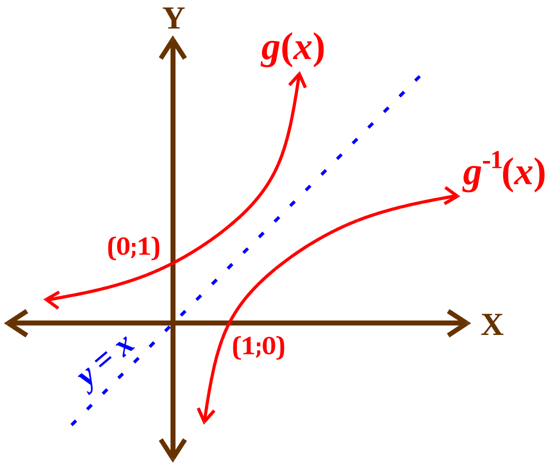
$\therefore g^{-1}(x) = \log_2 x$

(b) For $g(x)$:

x	-1	0	1
y	$\frac{1}{2}$	1	2

For $g^{-1}(x)$: x and y of $g(x)$ swap places:

x	$\frac{1}{2}$	1	2
y	-1	0	1



(c) Asymptote of $g^{-1}(x)$:

$x = 0$

<u>Exercise 3</u>:

(1) (a) Sketch: $g(x) = x^2 + 1$ for $x \leq 0$.

(b) Determine g^{-1} and write it in the form $g^{-1}(x) =$ …….

(c) Sketch $g^{-1}(x)$ on the same system of axes as $g(x)$.

(d) Write down the range of $g^{-1}(x)$.

(2) Given: $h(x) = 2^{-x}$

(a) Determine h^{-1} and write it in the form $h^{-1}(x) =$ …….

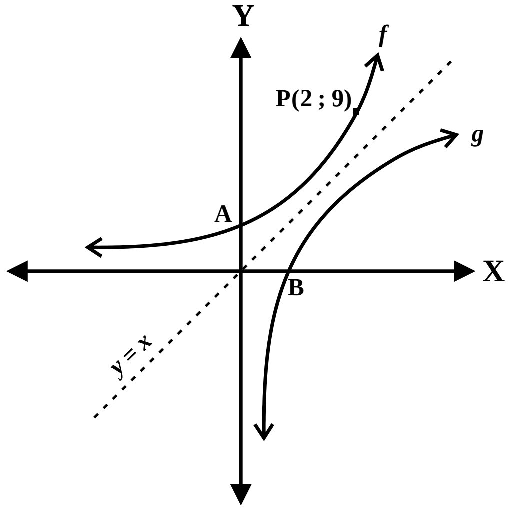
(b) Sketch h and h^{-1} on the same system of axes.

(c) Write down the domain of $h^{-1}(x)$.

(d) If p is the reflection of h in the *y*-axis, determine the equation of p and write it in the form $p(x) =$ …….

(e) Determine p^{-1} and write it in the form $p^{-1}(x) =$ …….

(3) Given: $f(x) = a^x$ and $g(x)$ with P$(2\ ;\ 9)$.



(a) Determine the value of a.

(b) Give the coordinates of A.

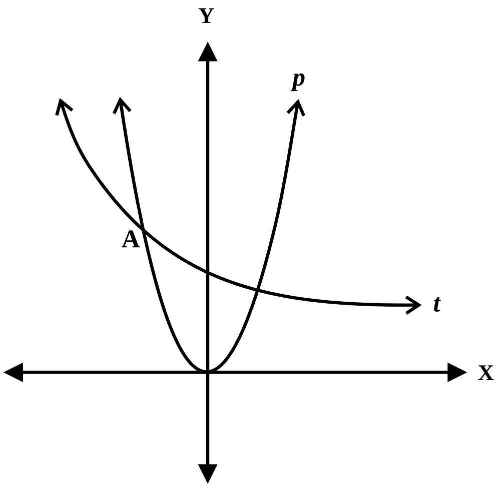
(c) Determine the equation of $g(x)$, if $g(x)$ is the mirror image of of $f(x)$ in the line $y = x$.

(d) Give the coordinates of B.

(e) For which values of *x* will $g(x)$ be defined?

(f) Write down the equation of the asymptote of $g(x)$.

(4) Given: $t(x) = a^x$ and $p(x) = bx^2$ met A$(-2\ ;\ 4)$.



(a) Determine the values of a and b.

(b) Write down the following: $t^{-1}(x) =$ ……

(c) Write down the following: $p^{-1}(x) =$ ……

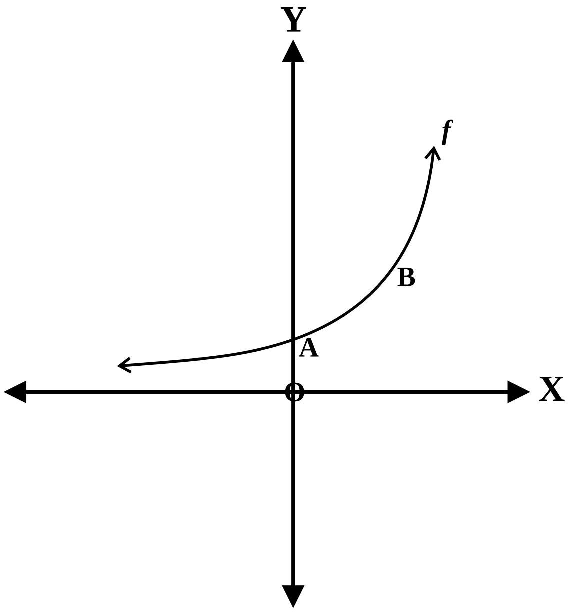
(d) Explain why $p^{-1}(x)$ is not a function.

(e) Determine x for which $t^{-1}(x) \geq 0$.

(f) Calculate: $t^{-1}(0{,}25) + p(3)$

(5) The graph of $f(x) = a^x$ is sketched alongside.

The point $\text{B}(3; 8)$ lies on the graph of f.



(a) Show that $a = 2$.

(b) Write down the coordinates of A.

(c) Write down the equation of $f^{-1}(x)$ in the form $f^{-1}(x) = \cdots$

(d) Sketch the graph of f^{-1}.
Show the x-intercept and ONE other point.

(e) For which values of x will $f^{-1}(x) = f(x)$?

(f) Write down the equation of g if g is the reflection of f in the y-axis.

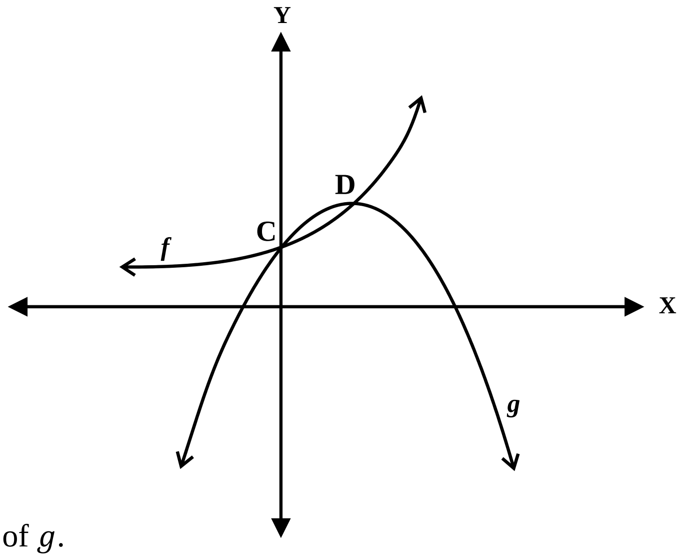
(g) Write down the equation of h if h is the reflection of f^{-1} in the x-axis.

(h) Are g and h one another's inverse? Motivate your answer.

(i) For which values of x will $f^{-1}(x) \geq 0$?

(j) Calculate: $f^{-1}(2) + f(-2)$

(6) On the right is the graphs of $f(x) = 2^x$ and $g(x) = -(x - 1)^2 + b$, with b as a constant value. The graphs of f and g intersects on the y-axis at C.
D is the turning point of g.



(a) Show that $b = 2$.

(b) Write down the coordinates of the turning point of g.

(c) Write down the equation of $f^{-1}(x)$ in the form $y = \ldots\ldots.$

(d) Sketch the graph of f^{-1} on the same graph as given above.
Show on your graph the x-intercept and the coordinates of one other point.

(e) Write down the equation of h if $h(x) = g(x + 1) - 2$.

(f) How can the domain of h be restricted so that h^{-1} will be a function?

(g) Determine the maximum value of $2^{2 - (x - 1)^2}$.

REVISION FROM PAST PAPERS:

Exercise A:

Consider the function $f(x) = \left(\frac{1}{3}\right)^x$

(1) Is f an increasing or decreasing function? Give a reason for your answer. (2)

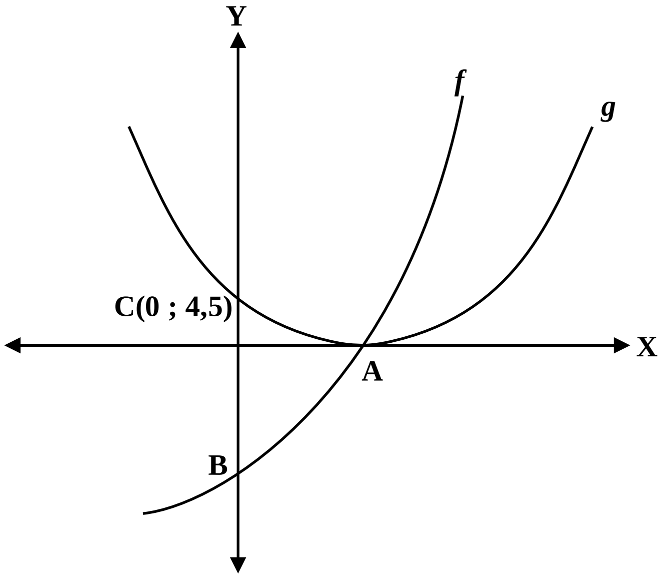
(2) Calculate $f^{-1}(x)$ in the form $y =$ …….. (2)

(3) Write down the equation of the asymptote of $f(x) - 5$. (1)

(4) Describe the transformation of f to g if $g(x) = \log_3 x$. (2)

Exercise B:

The graphs of $f(x) = 2^x - 8$ and $g(x) = ax^2 + bx + c$ sketched below. B and C$(0\ ;\ 4{,}5)$ are the y-intercepts of the graphs of f and g respectively. The two graphs intersect at A, which is the turning point of the graph of g and the x-intercept of the graphs of f and g.



(1) Determine the coordinates of A and B. (4)

(2) Write down the equation of the asymptote of graph f. (1)

(3) Determine the equation of h if $h(x) = f(2x) + 8$. (2)

(4) Determine the equation of h^{-1} in the form $y =$ ….. (2)

(5) Write down the equation of p, if p is the reflection of h^{-1} in the x-axis. (1)

(6) Calculate $\sum_{k=0}^{3} g(k) - \sum_{k=4}^{5} g(k)$. Show ALL calculations. (4)

Exercise C:

Given: $f(x) = 3^x$

(1) Determine an equation for f^{-1} in the form $f^{-1}(x) = ...$ (1)

(2) Sketch the graphs of f and f^{-1}, clearly showing ALL intercepts with the axes. (4)

(3) Write down the domain of f^{-1}. (2)

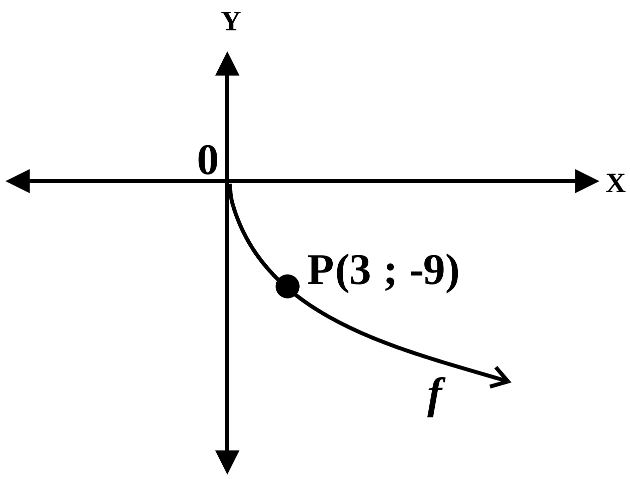
(4) For which values of x will $f(x).f^{-1}(x) \leq 0$? (2)

(5) Write down the range of $h(x) = 3^{-x} - 4$. (2)

(6) Write down an equation for g is the graph of g is the image of the graph of f after f has been translated two units to the right and reflected about the x-axis. (2)

Exercise D:

The graph of $f(x) = -\sqrt{27x}$ for $x \geq 0$ is sketched below.
The point $\mathrm{P}(3\,;\,-9)$ lies on the graph of f.



(1) Use the graph to determine the values of x for which $f(x) \geq -9$. (2)

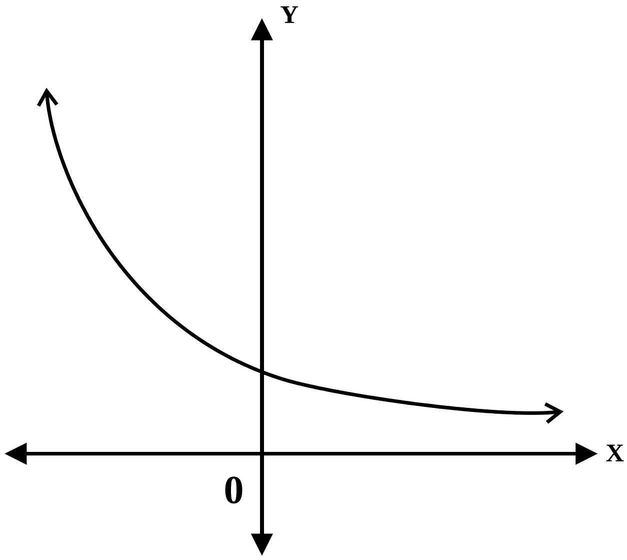
(2) Write down the equation of f^{-1} in the form $y = \ldots..$ (3)
Include ALL restrictions.

(3) Sketch f^{-1}, the inverse of f. Indicate the intercept(s) with the axes and the coordinates of ONE other point. (3)

(4) Describe the transformation of f to g if $g(x) = \sqrt{27x}$ for $x \geq 0$. (1)

Exercise E:

The graph of $f(x) = \left(\frac{1}{3}\right)^x$ is sketched on the right.



(1) Write down the domain of f. (1)

(2) Write down the equation of the asymptote of f. (1)

(3) Write down the equation of f^{-1} in the form $y = \ldots..$ (2)

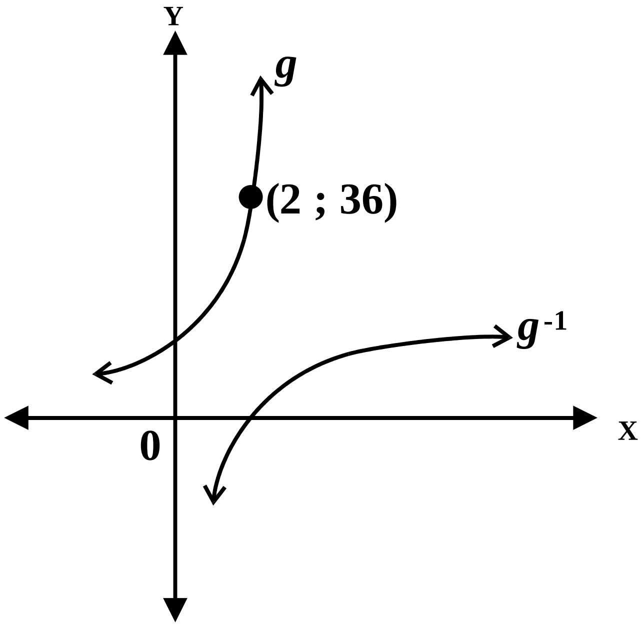
(4) Sketch the graph of f^{-1}. Indicate the x- intercept and the coordinates of ONE other point. (3)

(5) Write down the equation of the asymptote of $f^{-1}(x+2)$. (2)

(6) Prove that: $[f(x)]^2 - [f(-x)]^2 = f(2x) - f(-2x)$ for all values of x. (3)

Exercise F:

The graphs of $g(x) = k^x$, with $k > 0$ and $y = g^{-1}(x)$ is sketched on the right.
The point $(2\,;\,36)$ is a point on g.



(1) Determine the value of k. (2)

(2) Write down the equation of g^{-1} in the form $y = \ldots\ldots$ (2)

(3) For which value(s) of x will $g^{-1}(x) \leq 0$? (2)

(4) Write down the domain of h, for $h(x) = g^{-1}(x-3)$. (1)

(5) Sketch the graph of the inverse of $y = 1$. (2)

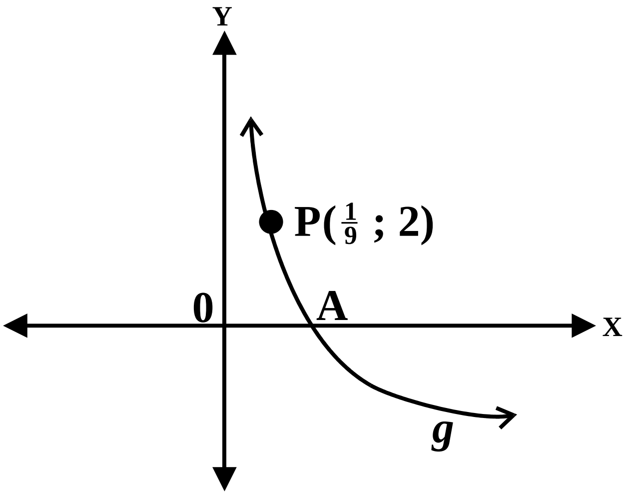
(6) Is the inverse of $y = 1$ a function? Motivate your answer. (2)

<u>Exercise G</u>:

Given the graph of $g(x) = \log_{\frac{1}{3}} x$

A is the x-intercept of g.

$P\left(\frac{1}{9}\ ;2\right)$ is a point on g.



(1) Write down the coordinates of A. (1)

(2) Sketch the graph of g^{-1} and indicate intercepts as well the coordinates of ONE other that will lie on the graph. (3)

(3) Write down the domain of g^{-1}. (1)