Grade 11 – Textbook

(CAPS Edition)

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Chapter A1

Number systems and exponents

A1.1 Number systems:

Exercise 1:

(1) Complete:

* Natural numbers: $\mathbb{N}$ = ______________________

* Whole numbers: $\mathbb{N}_0$ = ______________________

* Integers: $\mathbb{Z}$ = ______________________

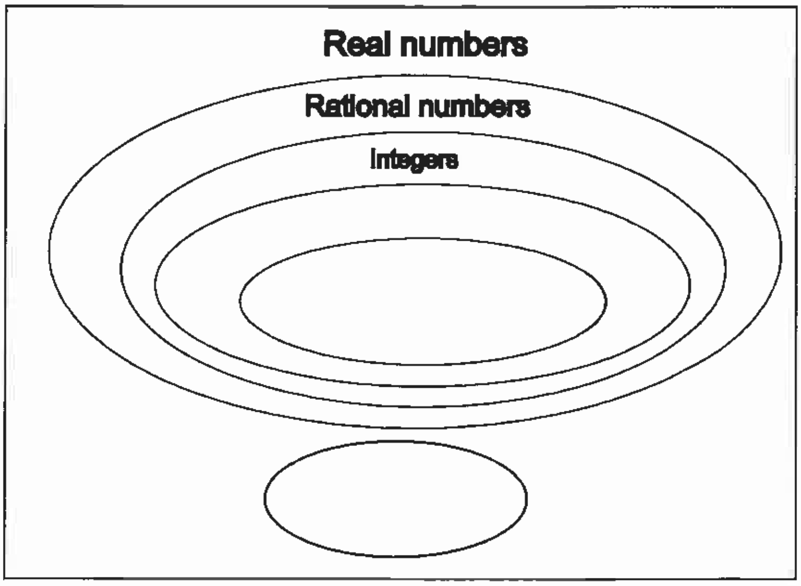
* Rational numbers: $\mathbb{Q}$ = ______________________

* Real numbers: $\mathbb{R}$ = ______________________

(2) Write three examples of Irrational numbers.

(3) Consider: $x(x - 6)(x^2 - 5)(2x^2 + x - 3) = 0$.
Solve for x and write the value(s) of x for which the solution of the equation will have:
(a) irrational roots.
(b) natural roots.
(c) integral roots.

(4) Complete the following diagram which presents the system of real numbers:



A1.2 Non-Real numbers:

Examples of non-real numbers: $\sqrt{-2}$; $\sqrt{-9}$ or $\sqrt[4]{-5}$
But not $\sqrt[3]{-8}$, because -2 × -2 × -2 = -8 $\therefore \sqrt[3]{-8}$ = -2

Exercise 2:

(1) Determine whether the following numbers are real or non-real. If real, indicate whether the number is rational or irrational:

(a) 7
(b) $-\sqrt{3}$
(c) π
(d) $\sqrt{-16}$
(e) $0,\dot{3}$
(f) $\frac{12}{36}$
(g) $\sqrt[3]{-125}$
(h) $1 + \sqrt{9}$
(i) $\sqrt{(-2)^3}$
(j) 0

(2) State whether the following statements are true or false:
(a) The product of two integers is always an integer again.
(b) The product of two irrational numbers is always an irrational number.
(c) If m is a natural number, $\sqrt{4}\,m$ will also be a natural number.
(d) The difference between two rational numbers is always a rational number.
(e) The quotient of a rational number and an irrational number will always be rational.

(3) For which values of x will the following statements be: (i) undefined (ii) non-real

(a) $\frac{x+3}{x}$
(b) $\sqrt{x-1}$
(c) $\frac{\sqrt{x}}{x+2}$

(4) Given: P = $\sqrt[4]{3y}$ - 1. To which of the following numbers system(s) will P belong if:
[Number systems: $\mathbb{N}$; $\mathbb{N}_0$; $\mathbb{Z}$; $\mathbb{Q}$; $\mathbb{Q}'$; $\mathbb{R}$ or $\mathbb{R}'$]
(a) $y = \frac{1}{3}$
(b) $y = -1$
(c) $y = 5$

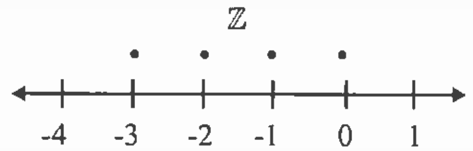
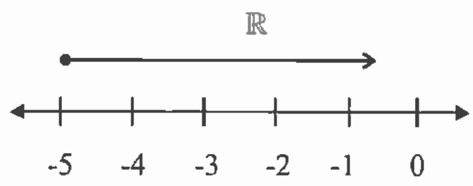
A1.3 Representation of real numbers:

As already seen in the previous grades, the real numbers can be represented by using on of the following ways:

(a) Interval notation.
(b) On a number line.
(c) As an inequality in set builder notation. Remember the following symbols:
$\cup \rightarrow$ the union of two or more intervals or sets.
$\cap \rightarrow$ the intersection of two or more intervals or sets.

Exercise 3:

Redraw and complete the following table:

	Set builder notation:	Interval notation:	Number line:
(1)	$\{x\,/\,-1 < x \leq 2\,;\, x \in \mathbb{R}\}$		
(2)		$x \in [-2\,;\,5]$	
(3)		$y \in (-\infty\,;\,3]$	
(4)			
(5)	$\{y\,/\,\ y \geq 3\,;\, y \in \mathbb{N}\}$		
(6)		$m \in (0\,;\,4]$	
(7)			
(8)	$\{m : m \leq 6\,;\, m \in \mathbb{R}\}$		
(9)	$\{x\,/\,-1 < x < 2\,;\, x \in \mathbb{Z}\}$		
(10)		$x \in (-1\,;\,\infty)$	

☺ **(1) Give a synonym for "non-real numbers".**

(2) Do some investigation regarding complex numbers and the numbers it contain.

A1.4 Exponents and surds:
A1.4.1 Exponents:

Basic exponential laws and properties:

(1) $x^m \times x^n = x^{m+n}$ (2) $x^m \div x^n = x^{m-n}$

(3) $(x^m)^n = x^{mn}$ (4) $(xy)^m = x^m y^m$ or $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$

(5) $x^0 = 1$ (6) $x^{-m} = \frac{1}{x^m}$

(7) $x^{\frac{m}{n}} = \sqrt[n]{x^m}$ *(m,n ∈ ℤ and n > 0 with n ≠ 1)*

E.g.1 *Simplify and write your answer as a positive exponent:*

(a) $$\frac{(x^3 \,.\, y^{-2})^2}{x^3(xy)^3} = \frac{x^6 \,.\, y^{-4}}{x^3 \,.\, x^3 \,.\, y^3} = \frac{x^6 \,.\, y^{-4}}{x^6 \,.\, y^3} = x^{6-6} \,.\, y^{-4-3} = x^0 \,.\, y^{-7} = \frac{x^0}{y^7} = \underline{\frac{1}{y^7}}$$

(b) $$\sqrt{x} \times x^{\frac{1}{3}} \div x^{\frac{1}{4}} = x^{\frac{1}{2}} \times x^{\frac{1}{3}} \div x^{\frac{1}{4}} = x^{\frac{1}{2}+\frac{1}{3}-\frac{1}{4}} = x^{\frac{6+4-3}{12}} = \underline{x^{\frac{7}{12}}}$$

(c) $$\frac{25^{n+1} \,.\, 10^n}{8^{n-1} \,.\, 5^{3n} \,.\, (2^{-1})^{2n}} = \frac{(5^2)^{n+1} \,.\, (2\times 5)^n}{(2^3)^{n-1} \,.\, 5^{3n} \,.\, 2^{-2n}}$$

$$= \frac{5^{2n+2} \,.\, 2^n \,.\, 5^n}{2^{3n-3} \,.\, 5^{3n} \,.\, 2^{-2n}} = \frac{5^{3n+2} \,.\, 2^n}{2^{n-3} \,.\, 5^{3n}}$$

$$= 5^{3n+2-3n} \,.\, 2^{n-(n-3)} = 5^2 \,.\, 2^{n-n+3} = \underline{5^2 \,.\, 2^3} = \underline{200}$$

(d) $$\frac{2^x - 2^{x+1}}{2^{x-1} + 2^x}$$

$$= \frac{2^x - 2^x \,.\, 2^1}{2^x \,.\, 2^{-1} + 2^x}$$

$$= \frac{2^x(1 - 2^1)}{2^x(2^{-1} + 1)} = \frac{1-2}{\frac{1}{2}+1} = -1 \div \left(\frac{1+2}{2}\right) = -1 \div \frac{3}{2} = -1 \times \frac{2}{3} = \underline{\frac{-2}{3}}$$

(e) $$\frac{x - x^{\frac{1}{2}} - 6}{x - 4} = \frac{(x^{\frac{1}{2}} - 3)\cancel{(x^{\frac{1}{2}} + 2)}}{(x^{\frac{1}{2}} - 2)\cancel{(x^{\frac{1}{2}} + 2)}} = \underline{\frac{x^{\frac{1}{2}} - 3}{x^{\frac{1}{2}} - 2}}$$

Exercise 4:

Simplify, without using a calculator: (Write answers as positive exponents!)

(1) $(125x^6)^{\frac{1}{3}}$ (2) $\left(x^{\frac{1}{2}} - 2\right)^2$

(3) $\sqrt[3]{-8x^9y^{-3}}$

(4) $3y^{\frac{1}{2}} \div (3y)^{\frac{-1}{2}}$

(5) $(0{,}25m^{\frac{1}{4}})^2$

(6) $\left(x^{\frac{1}{2}} + 4\right)\left(x^{\frac{1}{4}} - 2\right)\left(x^{\frac{1}{4}} + 2\right)$

(7) $\dfrac{x^{\frac{1}{2}} . \sqrt{x^3}}{x^{\frac{1}{3}}}$

(8) $\left(\dfrac{-12x^4y^4z}{-3x^2z^3}\right)^{\frac{1}{2}}$

(9) $\dfrac{m^{-2} - 3}{m^{-3} - 3m^{-1}}$

(10) $\dfrac{\left(9x^{\frac{2}{3}}y^{-4}\right)^{\frac{-3}{2}}}{3xy}$

(11) $\dfrac{(x + y)^{-1}}{x^{-1} - y^{-1}}$

(12) $\dfrac{2^{2n} - 3.2^n + 2}{2^n - 2}$

(13) $\left(m^{\frac{2}{3}} + n^{\frac{1}{3}}\right)^2$

(14) $\left(a^{\frac{1}{3}} - 5\right)\left(5 + a^{\frac{1}{3}}\right)$

(15) $\sqrt[3]{(0{,}125)^{-2}} + \left(125^2\right)^{\frac{1}{3}}$

(16) $\dfrac{12^{n+1} . 9^{n-2}}{18^{2n-1} . 3^{-n}}$

(17) $\dfrac{5^{n+1} . 25^{n-1}}{125^{n-2}}$

(18) $\dfrac{3^{2n} - 9^{n+1}}{3^{2n}}$

(19) $\dfrac{3 \times 2^x + 2^{x+1}}{5 \times 2^x}$

(20) $\dfrac{3^2 . 5^0 . 4^{n-1}}{2^{2n+1} - 2^{2n}}$

(21) $\dfrac{3^{-2x} . 36^{x+1} . 3}{4^{x-1} . (0{,}5)^2}$

(22) $\dfrac{5 . 5^{-y-1} + 5^{-2y} . 5^y}{3 . 5^{-y} - 5^{1-y}}$

A1.4.2 <u>Surds</u>:

Remember: $x^{\frac{m}{n}} = \sqrt[n]{x^m}$ $(m,n \in \mathbb{Z}$ and $n \geq 2)$

E.g.2 Simplify:

(a) $3\sqrt{2} + 7\sqrt{2} = \underline{10\sqrt{2}}$

(b) $\sqrt{8} \times \sqrt{2} = \sqrt{8 \times 2} = \sqrt{16} = \underline{4}$

(c) $\left(\sqrt{3} + 1\right)^2 = \left(\sqrt{3} + 1\right)\left(\sqrt{3} + 1\right) = 3 + 1\sqrt{3} + 1\sqrt{3} + 1$

$= \underline{4 + 2\sqrt{3}}$

(d) $\sqrt{18} + \sqrt{50} - 2\sqrt{8}$

$= \sqrt{9\times 2} + \sqrt{25\times 2} - 2\sqrt{4\times 2}$

$= \sqrt{9}\times\sqrt{2} + \sqrt{25}\times\sqrt{2} - 2\sqrt{4}\times\sqrt{2}$

$= 3\sqrt{2} + 5\sqrt{2} - 2\times 2\sqrt{2}$

$= 8\sqrt{2} - 4\sqrt{2}$

$= \underline{4\sqrt{2}}$

E.g.3 Rationalize the denominator: $\dfrac{2+\sqrt{8}}{\sqrt{2}}$

$$= \frac{2+\sqrt{8}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{2\times\sqrt{2} + \sqrt{8}\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}$$

$$= \frac{2\sqrt{2} + \sqrt{16}}{\sqrt{4}} = \frac{2\sqrt{2}+4}{2} = \frac{2\sqrt{2}}{2} + \frac{4}{2} = \underline{\sqrt{2} + 2}$$

Exercise 5:

(1) Simplify, without using a calculator:

(a) $(\sqrt{3} - 2)(\sqrt{3} + 2)$ (b) $\sqrt{8} + \sqrt{50} - \sqrt{18}$ (c) $\left(\sqrt{8} - 2^{\frac{1}{2}}\right)^2$

(d) $\sqrt[3]{27x^6} + \sqrt[5]{32x^{10}}$ (e) $(4\sqrt{2} - 3)^2$

(f) $m\times\sqrt{27m^6} - \sqrt{12m^8}$ (g) $\sqrt{3}\left(\sqrt{48} - 3\sqrt{75} + 2\sqrt{108}\right)$

(h) $\dfrac{\sqrt{18} - \sqrt{98}}{\sqrt{200}}$ (i) $\dfrac{\sqrt{\sqrt{64}} - \sqrt{12}}{\sqrt{18} - \sqrt{27}}$

(j) $\dfrac{(2+\sqrt{3})(4-\sqrt{3})}{\sqrt{100} + \sqrt{48}}$ (k) $\dfrac{\sqrt[3]{27x^6} + \sqrt[4]{16x^8}}{\sqrt[3]{125x^{27}}}$

(2) Simplify, without using a calculator. If necessary, rationalize the denominator.

(a) $\dfrac{\sqrt{3} + \sqrt{5}}{\sqrt{3}}$ (b) $\dfrac{\sqrt[3]{-8} - \sqrt[5]{32}}{\sqrt{72}}$ (c) $\dfrac{8\sqrt{5} - \sqrt{125}}{\sqrt{45}}$

A1.5 REVISION EXERCISE:

(1) Consider P = $\{-1\ ;\ \sqrt{8}\ ;\ \frac{1}{2}\ ;\ 101\ ;\ \sqrt[3]{125}\ ;\ 0{,}13526.....\ ;\ \frac{-3}{4}\ ;\ 5{,}6\dot{7}\ ;\ 0\ ;\ \frac{\sqrt{4}}{\sqrt{5}}\ ;\ \sqrt{-100}\ \}$

Write down the numbers in set P which belong to each of the following:

(a) Natural numbers (b) $\mathbb{Z}$

(c) Irrational numbers (d) $\mathbb{R}'$

(2) If $C = \sqrt{\dfrac{4-x}{-10x}}$, describe the value of C as real / non-real and rational / irrational if $x = -1$.

(3) Are the following true or false?

(a) $(-3)^2 = -3^2$ (b) $3^3 . 4^3 = 12^3$

(c) $8^x + 8^{2x} = 8^{3x}$ (d) $2^3 \times 2^5 = 4^4$

(e) $2^{-p} = \left(\frac{1}{2}\right)^p$

(4) Prove that $\sqrt{10}$ lies between 3 and 4, without using a calculator.

(5) Simplify the following, without using a calculator.

(a) $7m^{-2}n \times -2m^3n^4$ (b) $36^{\frac{1}{2}} \times 8^{\frac{2}{3}} + 64^{\frac{-1}{6}} \div 7^0$

(c) $\sqrt{\sqrt[3]{27}} \times \sqrt{12}$ (d) $\left(2\frac{1}{4}\right)^{\frac{1}{2}}$

(e) $\dfrac{15^{2n+1} . 9^{-n}}{25^n . 3^{1-n}}$ (f) $\left(\dfrac{9x^2y^{-4}}{z^6}\right)^{\frac{1}{2}} \div \dfrac{(8x^3)^{\frac{2}{3}}}{4(z^{-1}y^{-1})^3}$

(g) $\dfrac{2.5^m - 100.5^{m-2}}{5^m + 5^{m+1}}$ (h) $\dfrac{(0{,}25)^x \times 32^{x+1}}{24^x \times 3^{-x}}$

(i) $\dfrac{\sqrt{48y^{16}} + \sqrt{27}\ y^{16}}{4 + 3y^8}$ (j) $\dfrac{3m.3^{3x-1} + m.27^x}{m^4 . 3^{4x} . (3^{-1})^x . m^{-3}}$

(6) If $\sqrt{2} = m$ and $\sqrt[3]{3} = n$, prove that: $\sqrt{32} + \sqrt{18} + \sqrt[3]{6} \times \sqrt[3]{4} - \sqrt{72} = m + 2n$

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Chapter A2

Algebraic expressions and equations

A2.1 Simplifying algebraic fractions:

E.g.1 Simplify:

(a) $\dfrac{2x^2 - 4x - 16}{4 - x^2}$ (b) $\dfrac{(y + 2)^2 - y - 2}{y + 1}$

****

(a)

$$= \frac{2(x^2 - 2x - 8)}{-(x^2 - 4)}$$

$$= \frac{2(x - 4)\cancel{(x + 2)}}{-(x - 2)\cancel{(x + 2)}}$$

$$= \frac{2(x - 4)(x + 2)}{-(x - 2)(x + 2)}$$

$$= \frac{-2(x - 4)}{(x - 2)}$$

(b)

$$= \frac{(y + 2)^2 - 1(y + 2)}{(y + 1)}$$

$$= \frac{(y + 2)(y + 2 - 1)}{(y + 1)}$$

$$= \frac{(y + 2)\cancel{(y + 1)}}{\cancel{(y + 1)}}$$

$$= \underline{y + 2}$$

Exercise 1:

Simplify: (No denominators are equal to zero!)

(1) $\dfrac{x^2 - 5x + 6}{x^2 - 9}$

(2) $\dfrac{2}{x+1} - \dfrac{2}{x-1}$

(3) $\dfrac{m^2 + 10m + 25}{2m^2 + 10m} \times \dfrac{m^2 - m}{m^2 + 4m - 5}$

(4) $\dfrac{p}{p^2 - 1} + \dfrac{3}{p^2 - p - 2}$

(5) $\dfrac{2(x-1)}{x^2 - 4} - \dfrac{3}{6 - x - x^2}$

(6) $\dfrac{y^3 - 4y}{6y^2} \div \dfrac{y^2 - 2y - 8}{y^2 - 4y}$

(7) $\dfrac{p^2 + 9}{p + 3} \times \dfrac{(p - 3)^2}{p} \div \dfrac{p^4 - 81}{1}$

(8) $\dfrac{2}{m^2 + 3m + 2} + \dfrac{m}{m^2 - 4} + 3$

A2.2 Equations with fractions:

*** The difference between an expression and an equation:**

Algebraic expression:

$$\frac{x-1}{x^2-4}+\frac{2}{x^2-2x}$$

$$\therefore \frac{x-1}{(x+2)(x-2)}+\frac{2}{x(x-2)}$$

LCM = $x(x+2)(x-2)$

$$\therefore \frac{x(x-1)+2(x+2)}{x(x+2)(x-2)}$$

$$\therefore \frac{x^2-x+2x+4}{x(x+2)(x-2)}$$

$$\therefore \underline{\frac{x^2+x+4}{x(x+2)(x-2)}}$$

Equation:

$$\frac{x-1}{x^2-4}=\frac{2}{x^2-2x}$$

$$\therefore \frac{x-1}{(x+2)(x-2)}=\frac{2}{x(x-2)}$$

$\therefore\ x \neq 0$; $x \neq -2$ and $x \neq 2$

$$\therefore \frac{(x-1)}{\cancel{(x+2)}\cancel{(x-2)}}\times\frac{x\cancel{(x+2)}\cancel{(x-2)}}{1}=\frac{2}{\cancel{x}\cancel{(x-2)}}\times\frac{\cancel{x}(x+2)\cancel{(x-2)}}{1}$$

$$\therefore x(x-1)=2(x+2)$$

$$\therefore x^2-x=2x+4$$

$$\therefore x^2-3x-4=0$$

$$\therefore (x-4)(x+1)=0$$

$$\therefore \underline{x=4} \quad \text{or} \quad \underline{x=-1}$$

E.g. 2 Solve x:

$$\frac{2x-3}{x^2+6x+8}-\frac{x-4}{x^2+3x-4}=\frac{3x+8}{x^2+x-2}$$

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$$\frac{2x-3}{(x+2)(x+4)}-\frac{x-4}{(x+4)(x-1)}=\frac{3x+8}{(x+2)(x-1)}$$

LCM = $(x+2)(x+4)(x-1)$ $\therefore\ x \neq -2$; $x \neq -4$ and $x \neq 1$

$$\frac{(2x-3)}{\cancel{(x+2)}\cancel{(x+4)}}\times\frac{\cancel{(x+4)}\cancel{(x+2)}(x-1)}{1}-\frac{(x-4)}{\cancel{(x+4)}\cancel{(x-1)}}\times\frac{\cancel{(x+4)}(x+2)\cancel{(x-1)}}{1}=\frac{(3x+8)}{\cancel{(x+2)}\cancel{(x-1)}}\times\frac{(x+4)\cancel{(x+2)}\cancel{(x-1)}}{1}$$

$$\therefore (2x-3)(x-1)-(x-4)(x+2)=(3x+8)(x+4)$$

$$2x^2-2x-3x+3-(x^2+2x-4x-8)=3x^2+12x+8x+32$$

$$2x^2-5x+3-x^2-2x+4x+8=3x^2+20x+32$$

$$x^2-3x+11=3x^2+20x+32$$

$$0=3x^2+20x+32-x^2+3x-11$$

$$0=2x^2+23x+21$$

$$0=(2x+21)(x+1)$$

$$2x+21=0 \quad \text{or} \quad x+1=0$$

$$\underline{x=\tfrac{-21}{2}} \qquad \underline{x=-1}$$

Exercise 2:
Solve the following equations:

(1) $x = \dfrac{5}{x - 4}$

(2) $\dfrac{y - 2}{y - 1} = \dfrac{2y - 1}{y + 7}$

(3) $\dfrac{m}{m + 1} = \dfrac{m - 2}{m + 3}$

(4) $1 - \dfrac{1}{p - 1} = \dfrac{p - 3}{p - 1}$

(5) $\dfrac{4}{x^2 - 4} - \dfrac{10}{x^2 - x - 6} = \dfrac{1}{x + 2}$

(6) $\dfrac{10}{y^2 - 2y - 8} + \dfrac{5}{y + 2} = -1$

(7) $\dfrac{y}{y - 1} = 2 + \dfrac{2}{1 - y} + \dfrac{2}{y + 1}$

(8) $\dfrac{6}{m^2 - 9} - \dfrac{1}{3 - m} = \dfrac{2m}{m + 3}$

(9) $\dfrac{3 - x}{x^2 + 6x + 5} = \dfrac{3}{x^2 + x} - \dfrac{2}{5 + x}$

(10) $\dfrac{2x}{3x - 6} - 1 = \dfrac{2(x + 1)}{x^2 - 4} - \dfrac{1}{x + 2}$

A2.3 Solving equations by using squaring and extraction of roots:

E.g.3 Solve x:

(a) $2(x + 3)^2 - 8 = 0$

(b) $2\sqrt{x + 1} + x - 2 = 0$

****

(a)
$$2(x + 3)^2 = 8$$
$$(x + 3)^2 = \tfrac{8}{2}$$
$$(x + 3)^2 = 4$$
$$x + 3 = \pm\sqrt{4}$$
$$x + 3 = \pm 2$$
$x + 3 = 2$ or $x + 3 = -2$

$\underline{x = -1}$ $\qquad$ $\underline{x = -5}$

(b)
$$2\sqrt{x + 1} = 2 - x$$
$$\left(2\sqrt{x + 1}\right)^2 = (2 - x)^2$$
$$(2)^2\left(\sqrt{x + 1}\right)^2 = 4 - 4x + x^2$$
$$4(x + 1) = x^2 - 4x + 4$$
$$4x + 4 = x^2 - 4x + 4$$
$$0 = x^2 - 8x$$
$$0 = x(x - 8)$$
$x = 0$ or $x = 8$

CHECK:

*$LHS = 2\sqrt{0 + 1} + 0 - 2$ $\qquad$ $RHS = 0$

$= 2(1) - 2 = 0$

$\therefore$ $LHS = RHS$ $\quad\therefore$ $\underline{x = 0}$

*$LHS = 2\sqrt{8 + 1} + 8 - 2$ $\qquad$ $RHS = 0$

$= 2(3) - 6 = 0$

$\therefore$ $LHS = RHS$ $\quad\therefore$ $\underline{x = 8}$

Exercise 3:

Solve x: (if necessary, leave your answer in simplest surd form.)

(1) $\sqrt{2x+3} = 4$

(2) $(x-1)^2 - 4 = 0$

(3) $\sqrt{x+12} - x = 0$

(4) $\sqrt{x+1} = x - 1$

(5) $x + 1 = \sqrt{2x+5}$

(6) $x - \sqrt{x} = 2$

(7) $\sqrt{5-3x} = \sqrt{x+1}$

(8) $4(x+3)^2 - 16 = 9$

(9) $x - \sqrt{3-2x} = 0$

(10) $2\sqrt{1-x} + 1 = x$

A2.4 K-method / Substitution:

E.g.4 Solve for x: $(x^2 - x)^2 - 8(x^2 - x) + 12 = 0$

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Our first reaction is usually to remove the brackets:

$x^4 - 2x^3 + x^2 - 8x^2 + 8x + 12 = 0$

But then we are left with an equation to the fourth degree to solve! You can't do this yet!
Therefore we take another route namely substitution or also known as the k-method:
Substitute each $(x^2 - x)$ *with a* k: $\therefore$ *Let* $(x^2 - x) = k$

$x^4 - 2x^3 + x^2 - 8x^2 + 8x + 12 = 0$

$\therefore\ k^2 - 8k + 12 = 0$

$\therefore\ (k-6)(k-2) = 0$

$\therefore\ k - 6 = 0$ *or* $k - 2 = 0$

$\therefore\ x^2 - x - 6 = 0$ *or* $x^2 - x - 2 = 0$ *[substitute k with* $(x^2 - x)$*]*

$\therefore\ (x-3)(x+2) = 0$ $\quad$ $(x-2)(x+1) = 0$

$\therefore\ \underline{x = 3}$ *or* $\underline{x = -2}$ $\quad$ $\underline{x = 2}$ *or* $\underline{x = -1}$

Exercise 4:

Solve the following equations:

(1) $(y^2 - 3y)^2 - 2(y^2 - 3y) = 8$

(2) $(x^2 - 5x)^2 = 36$

(3) $\dfrac{1}{x^2 - x - 1} = x^2 - x - 1$

(4) $4(m^2 - m) - 7 = \dfrac{2}{m^2 - m}$

(5) $\sqrt{x-3} = 2 - \dfrac{1}{\sqrt{x-3}}$

(6) $x^2 - 5x + 3 - \dfrac{9}{x^2 - 5x + 3} = 0$

(7) $(y^2 - 2y)^2 - 2y^2 + 4y - 3 = 0$

(8) $x^2 + x + 2 = \dfrac{-8}{x^2 + x - 4}$

A2.5 Completing the square:

The following are examples of complete squares:
16 ; x^2 ; $(-y)^2$; $(x - 2)^2$; $(p + 5)^2$ etc.

$(x^2 - 8x + 16)$ is also a complete square because
$(x^2 - 8x + 16) = (x - 4)(x - 4) = (x - 4)^2$

$\therefore$ Any expression in the format $x^2 + bx$ can be written as a complete square by adding the correct constant value (c): $c = \left(\frac{b}{2}\right)^2$

E.g. $x^2 - 6x \rightarrow x^2 - 6x + \left(\frac{-6}{2}\right)^2 \rightarrow x^2 - 6x + \mathbf{9} = (x - 3)^2 \rightarrow$ complete square!
or $x^2 + 9x \rightarrow x^2 + 9x + \left(\frac{9}{2}\right)^2 \rightarrow x^2 - 6x + \mathbf{20{,}25} = (x + \frac{9}{2})^2 \rightarrow$ complete square!

E.g.5 Solve x by completing the square:

(a) $x^2 - 4x - 12 = 0$ *(b)* $2x^2 + 3x + 1 = 0$

****

(a)

$x^2 - 4x = 12$

$x^2 - 4x + \left(\frac{4}{2}\right)^2 = 12 + \left(\frac{4}{2}\right)^2$

$x^2 - 4x + 4 = 12 + 4$

$(x - 2)^2 = 16$

$\sqrt{(x-2)^2} = \pm\sqrt{16}$

$x - 2 = \pm 4$

$\therefore x - 2 = 4$ or $x - 2 = -4$

$\underline{x = 6}$ $\underline{x = -2}$

(b)

$2x^2 + 3x = -1$

$\frac{2x^2}{2} + \frac{3x}{2} = \frac{-1}{2}$

$x^2 + \frac{3}{2}x = \frac{-1}{2}$

$x^2 + \frac{3}{2}x + \left(\frac{3}{2} \div 2\right)^2 = \frac{-1}{2} + \left(\frac{3}{2} \div 2\right)^2$

$x^2 + \frac{3}{2}x + \left(\frac{3}{2} \times \frac{1}{2}\right)^2 = \frac{-1}{2} + \left(\frac{3}{2} \times \frac{1}{2}\right)^2$

$x^2 + \frac{3}{2}x + \left(\frac{3}{4}\right)^2 = \frac{-1}{2} + \left(\frac{3}{4}\right)^2$

$\left(x + \frac{3}{4}\right)^2 = \frac{-1}{2} + \frac{9}{16}$

$\left(x + \frac{3}{4}\right)^2 = \frac{1}{16}$

$x + \frac{3}{4} = \pm\frac{1}{4}$

$\therefore x + \frac{3}{4} = \frac{1}{4}$ or $x + \frac{3}{4} = -\frac{1}{4}$

$x = -\frac{3}{4} + \frac{1}{4}$ $x = -\frac{3}{4} - \frac{1}{4}$

$\underline{\mathbf{x} = \frac{-2}{4} = -\frac{1}{2}}$ $\underline{\mathbf{x} = \frac{-4}{4} = -1}$

Exercise 5:

(1) Solve the unknown in each of the following by completing the square:
[If necessary, leave the answer in simplest surd form.]

(a) $x^2 + 8x - 9 = 0$ (b) $p^2 - 12p + 32 = 0$

(c) $x^2 + 2x + 7 = 0$ (d) $x^2 = 3x + 5$

(e) $(m - 4)(m + 2) = 6$ (f) $2y^2 + 12y + 2 = 0$

(2) Solve the unknown in each of the following by completing the square:
[If necessary, round off correct to one decimal.]

(a) $2x(x - 10) = 4$ (b) $p^2 - 2p + 3 = 2p$

(c) $m^2 - 3m + 1 = 2m^2$ (d) $(2x + 1)(x - 1) = 2$

(3) Solve x by completing the square:

(a) $5x^2 - 15x + 30 = 0$ (b) $ax^2 + 2ax - 3 = 0$

(c) $(x - p)(x + p) = 2px$ (d) $ax^2 + bx + c = 0$

A2.6 Quadratic formula:

Consider ex.5 no.3d: If x is solved in $ax^2 + bx + c = 0$ in terms of a, b and c, the following will be the answer:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The above formulae is the formulae used to solve any quadratic equation as written in the format , $ax^2 + bx + c = 0$.

Eg.6 Solve x: $2(x - 3)(x + 1) = 3$
[If necessary, leave the answer in simplest surd form.]

****

$2(x - 3)(x + 1) = 3$

$2(x^2 - 2x - 3) = 3$

$2x^2 - 4x - 6 - 3 = 0$

$2x^2 - 4x - 9 = 0$ $\therefore\ a = 2\ ;\ b = -4$ *and* $c = -9$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-9)}}{2(2)}$$

$$= \frac{4 \pm \sqrt{16 + 72}}{4}$$

$$= \frac{4 \pm \sqrt{88}}{4}$$

$$x = \frac{4}{4} \pm \frac{2\sqrt{22}}{4}$$

$$\therefore \quad \boldsymbol{x = 1 \pm \frac{\sqrt{22}}{2}}$$

Exercise 6:

(1) Solve x: [If necessary, round off, correct to 2 decimals.]

(a) $3x^2 + 4x - 5 = 0$ (b) $1 + 2x - x^2 = 0$

(c) $(x - 3)(2x + 1) = 2$ (d) $4x + 1 = x^2 - 7$

(e) $2x^2 + 5x - 7 = 0$ (f) $x^2 - 6x = 1$

(g) $(x + 1)^2 - 2(x + 1)(x - 1) = 7$ (h) $px^2 - 3px + 2 = 0$

(2) Solve x: [If necessary, leave the answer in simplest surd form.]

(a) $\dfrac{2}{x^2 - 1} + \dfrac{x}{x^2 - x} = \dfrac{3}{x}$ (b) $\dfrac{3}{x} - \dfrac{x + 1}{x^2 + 2x} = \dfrac{2}{x^2 - 4}$

(3) (a) Solve for m: $m - \dfrac{8}{m} = 2$

(b) Hence, or in any other way, solve x: $x^2 + 2x + 1 - \dfrac{8}{x^2 + 2x + 1} = 2$

[If necessary, round off to 1 decimal.]

(4) (a) Solve for p: $\dfrac{6}{p} = p - 1$

(b) Hence, or in any other way, solve x: $\dfrac{6}{x(x + 2)} = x^2 + 2x - 1$

[If necessary, round off to the nearest integer.]

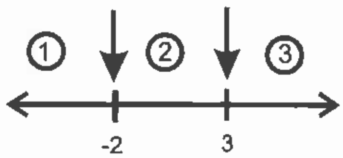
A2.7 Quadratic inequalities:

Method 1:

E.g.7 Solve x: $(x - 3)(x + 2) \leq 0$

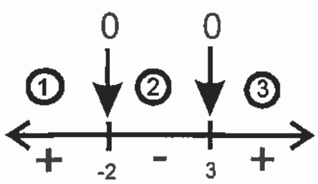
****

Let $y = (x - 3)(x + 2)$
Investigate the possible outcomes in the different areas by randomly choosing a possible value for x. To determine the outcome:



Choose any value for x in area 1:	$y = (-4 - 3)(-4 + 2) = (-7)(-2)$	$= +14$
Choose $x = -2$:	$y = (-2 - 3)(-2 + 2) = (-5)(0)$	$= 0$
Choose any value for x in area 2:	$y = (0 - 3)(0 + 2) = (-3)(2)$	$= -6$
Choose $x = 3$:	$y = (3 - 3)(3 + 2) = (0)(5)$	$= 0$
Choose any value for x in area 3:	$y = (7 - 3)(7 + 2) = (4)(9)$	$= +36$

$\therefore$ *If* $(x - 3)(x + 2) \leq 0$, *we obtain the solution from the negative area, but also where the solution is equal to 0:*



$\therefore$ $\underline{-2 \leq x \leq 3}$

Method 2:

E.g.8 Solve x: $x^2 > 4$

****

$x^2 - 4 > 0$ *[The one side must be equal to 0.]*

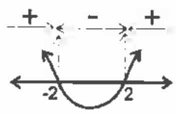
$(x - 2)(x + 2) > 0$ *[Factorise.]*

Now we use the quadratic function (parabola) as sketched in grade 10!

Remember:

If $a > 0$:  *and if* $a < 0$: 

$\therefore$ *Make a rough sketch of* $y = x^2 - 4$.
Show the form and the x-intercepts.



$\therefore$ *Solution:* $\underline{x < -2 \text{ of } x > 2}$

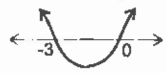
E.g.9 Solve x: $-x^2 - 3x \leq 0$

****

Divide throughout with -1: $x^2 + 3x \geq 0$

Factorise: $x(x + 3) \geq 0$

Interpret the solution graphically:



Solution: $\underline{\boldsymbol{x \leq -3 \text{ or } x \geq 0}}$

E.g.10 Solve m: $2m^2 + 2 > -4m$

****

$\therefore \quad 2m^2 + 4m + 2 > 0$ *[Get all terms on the one side]*

$\therefore \quad m^2 + 2m + 1 > 0$ *[Divide throughout with 2]*

$\therefore \quad (m + 1)(m + 1) > 0$ *[Factorise]*

$\therefore \quad (m + 1)^2 > 0$

But any complete square, $(\ )^2$, *is always positive!*

$\therefore \quad \underline{\boldsymbol{m \in R, \text{ but } m \neq -1}}$ $[(m + 1)^2 \neq 0$, *because it is only >]*

Exercise 7:

(1) Solve the following inequalities:

(a) $x^2 - 7x + 12 > 0$ (b) $p^2 \geq 25$

(c) $3n^2 - 4n > 0$ (d) $4 - 3x < x^2$

(e) $m^2 \leq -2m + 8$ (f) $(x - 1)(x + 1) > 35$

(g) $25 - 10x + x^2 \geq 0$ (h) $(m + 3)(m - 1) < 0$

(i) $x^2 - 7x \leq 3x - 16$ (j) $2p(p + 3) > 20$

(k) $(x - 1)(x + 2)(x + 5) \leq 0$ (l) $7x + 3 \leq 6x^2$

(m) $2x^2 - 3x + 4 < 3(1 - x)$ (n) $(p + 1)^3 - 4(p + 1) > 0$

(o) $2(x + 1)^2 - 11 < (x - 2)(x + 2)$ (p) $x^3 + 2x^2 + x > 0$

(2) Consider: $y = \dfrac{-2}{x + 2} - \sqrt{x^2 - 9}$

(a) For which values of x will y be undefined?

(b) For which values of x will y be non-real?

(3) Consider: $y = x^2 + 4x + 5$

(a) Solve x if $y = 0$

(b) Write $y = x^2 + 4x + 5$ in the form $y = (x + p)^2 + q$ by completing the square.

(c) Hence, or any other way, solve x: $x^2 + 4x + 5 > 0$

(4) (a) Solve x: $(x + 1)^2 - 3(2x + 2) + 1 = 0$
[If necessary, leave the answer in simplest surd form.]

(b) Hence, solve x: $(x + 1)^2 < 3(2x + 2) - 1$

A2.8 Simultaneous equations:

E.g.11 Solve x and y: $2x - y = 1$ *and* $2x^2 = 3 - y^2$

****

From: $2x - y = 1$

$2x - 1 = y$ → *substitute* → $2x^2 = 3 - y^2$

$2x^2 = 3 - (2x - 1)^2$

$2x^2 = 3 - (4x^2 - 4x + 1)$

$2x^2 = 3 - 4x^2 + 4x - 1$

$6x^2 - 4x - 2 = 0$

$3x^2 - 2x - 1 = 0$

$(3x + 1)(x - 1) = 0$

$3x + 1 = 0$ *or* $x - 1 = 0$

$\therefore \underline{x = \frac{-1}{3}}$ $\qquad \underline{x = 1}$

$2(\frac{-1}{3}) - 1 = y$

$\frac{-2}{3} - 1 = y$

$\therefore \underline{y = -1\frac{1}{3}}$

$2(1) - 1 = y$

$\therefore \underline{y = 1}$

Exercise 8:

(1) Solve the following equations simultaneously:

(a) $y - x = 1$ and $2xy - x^2 + 1 = 3y$

(b) $m + n - 3 = 0$ and $m^2 - 2m + 3n^2 = 2$

(c) $2a - b = -1$ and $3a^2 - 2b - b^2 = 4$

(d) $p - 4 = 2q$ and $p^2 - 3pq + 2q^2 - 4 = 0$

(e)	$y + 2x = 3$	and	$y^2 - 2xy - y = 0$
(f)	$2a = 3b$	and	$2a^2 - 4a + b^2 = 10$
(g)	$xy = -4$	and	$x(x + 1) + y - 3x - x^2 = -3(x + 1)$
(h)	$\frac{x-1}{2} = y$	and	$x^2 = 2xy - y^2 + 1$
(i)	$2m - n = 3$	and	$m^2 + n^2 = 5$
(j)	$2x - 3y = 4$	and	$(2x - 3y)(x - 2y) = 12$

(2) (a) Solve for p and q: $p + q = 3$ and $p^2 - pq + q^2 = 3$

(b) Hence or any other way, solve for x and y: $\frac{1}{x} + \frac{1}{y} = 3$ and $\frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2} = 3$

A2.9 Exponential equations:

E.g.12 Solve x: (a)

$$27^x = \frac{1}{9}$$

$$\left(3^3\right)^x = \frac{1}{3^2}$$

$$3^{3x} = 3^{-2}$$

$$EB \Leftrightarrow EE$$

$$3x = -2$$

$$\underline{x = \tfrac{-2}{3}}$$

(b)

$$2x^{\frac{1}{3}} = 6$$

$$x^{\frac{1}{3}} = 3$$

$$\left(x^{\frac{1}{3}}\right)^{\frac{3}{1}} = (3)^3$$

$$\underline{x = 27}$$

E.g.13 Solve x:

$$2^x = 2^{x+2} - 12$$

$$2^x - 2^{x+2} = -12$$

$$2^x - 2^x . 2^2 = -12$$

$$2^x(1 - 4) = -12$$

$$2^x(-3) = -12$$

$$2^x = \tfrac{-12}{-3}$$

$$2^x = 4$$

$$2^x = 2^2$$

$$\therefore \underline{x = 2}$$

<u>Exercise 9</u>:

Solve for x:

(1) $2^{x} = 64$ (2) $3^{1-x} - 27 = 0$ (3) $4^{x} = 0{,}5$

(4) $2 \times 3^{x+3} = 18$ (5) $\frac{1}{8} = 4^{2x}$ (6) $25^{2x+1} - 1 = 0$

(7) $3^{3x} \cdot 3^{1-x} = 27$ (8) $\sqrt{2} \cdot 2^{2-x} = 2$ (9) $2x^{\frac{1}{2}} = 16$

(10) $3^{\frac{2}{x}} = 27$ (11) $4^{1-x} \times 8^{x} = 1$ (12) $125^{x} = 0{,}04$

(13) $5^{x-1} + 5^{x} = 30$ (14) $2^{2x+1} - 4^{x-1} - 14 = 0$

(15) $3^{x+2} - 2^{3} = 3^{x+1} + 10$ (16) $(3 \times 2^{x+1})^{2} = 9 \times 4^{1-x}$

(17) $2^{x-2} = 5 - 2^{x}$ (18) $(9^{x+1} - 27)(9^{x} + 3^{2x+1} - 12) = 0$

A2.10 REVISION EXERCISE:

(1) Consider: $A = \dfrac{1}{1-2x} - \dfrac{2-8x}{4x^{2}-1}$

(a) Simplify A
(b) For which value(s) of x will A be undefined.
(c) Solve for x if $A = 2$.

(2) (a) Solve for p: $p + \frac{12}{p} = -7$

(b) Hence, or in any other way, solve x: $2x^{2} - 3x - 3 + \dfrac{12}{2x^{2} - 3x - 3} = -7$

(3) Consider: $B = \sqrt{2x^{2} + x - 3} + x$

(a) For which values of x will B be non-real?

(b) Calculate x if $B = 1$.

(4) Solve x by using **two** different methods: $2x^{2} - 8x + 4 = 0$
If necessary, leave your answer in simplest surd form.

(5) Determine two values for m for which the following expressions will be complete squares:

(a) $x^{2} - mx + 10$ (b) $y^{2} + m(2y - 3) + 4$

(6) Determine x in terms of m and n by completing the square:

$$2mx^{2} - 4mx + 6n = 0$$

(7) Consider: $(2x - 1)(y + 4) = 0$. For which values of y will:

(a) $x = -1$ (b) $x = \frac{1}{2}$

(8) Solve x. If necessary, approximate to three decimals:

(a) $x(x - 3) + 4 = 0$ (b) $x - 3x^2 = -4$

(c) $\dfrac{x + 2}{x + 3} = \dfrac{x}{x + 1}$ (d) $3^x + 3^{x+2} = 10$

(e) $\dfrac{3}{x^2 - x} - \dfrac{1}{x^2 - 1} + \dfrac{3}{x} = 0$ (f) $27x^{\frac{3}{2}} = 64x^{\frac{-3}{2}}$

(9) Solve the following equations simultaneously:

(a) $x - y = 5$ and $x^2 + 2y^2 = 57$

(b) $2x - y = 3$ and $x^2 + xy - y^2 = -19$

(10) (a) Solve x, correct to one decimal:
$x(x - 3) - 3(x + 1) = 1$

(b) Hence, or in any other way, solve x:
$x^2 - 6x - 4 > 0$

(11) If $f(x) = x^2 + 6x$ and $g(x) = 8 - x^2$, calculate:
[If necessary, leave your answer in simplest surd form.]

(a) x if $g(x) = 0$ (b) $f(-6)$

(c) x if $f(x) = g(x)$:

(12) Consider: $P = \dfrac{\sqrt{9 - x^2}}{x^2 + x}$. For which value(s) of x will:

(a) P be non-real (b) P be undefined

(13) Consider: $h(x) = x(2x + 1)(x^2 + 3)(x - 6)$. Solve x if $h(x) = 0$, with x:

(a) a whole number (b) an irrational number.
(c) non-real. (d) an integer.

(14) Determine $\frac{y}{x}$ if $4x^2 - 4xy - 3y^2 = 0$

(15) (a) Prove that $3(x + 1)^2 + 4 > 0$ for all real values of x.

(b) Hence, or in any other way, solve x: $\dfrac{x^2 - 4}{3(x + 1)^2 + 4} \leq 0$

(16) If one of the roots of $mx^2 - 4x + 1 = 0$, is equal to 1, calculate the value of m as well as the other root.

(17) $p(x) = x^2 + 4x$ and $q(x) = p(x) + 5$

(a) Prove, by completing the square that $q(x) = (x + 2)^2 + 1$
(b) Hence, or in any other way, solve x if $q(x) \geq 0$

****